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Perturbation Treatment for the Vibrations of a Circular Membrane Subject to a Restorative Force

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Abstract
We have investigated the motion of a stretched elastic circular membrane which is subjected to a restorative force proportional to the velocity. The wave equation which describes the vibrations in this case is written in polar form. The perturbation method is applied together with the usual method of separation of variables. The oscillations are then given at any instant of time t and point on the membrane depending on the given boundary and initial conditions. The roles of the restorative force and the initial displacement are discussed. The numerical solutions are then given by using the program Mathematica.

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1. Introduction
Most of the physically important partial differential equations are of second order and can be classified into three types: elliptic, parabolic, and hyperbolic. Elliptic partial differential equations involve second order derivatives in each of the independent variables, each derivative having the same sign when all terms in the equation are grouped on one side. Roughly speaking, parabolic partial differential equations involve only a first-order derivative in one variable, but have second order derivatives in the remaining variables. Hyperbolic partial differential equations involve second-order derivatives of opposite sign, such as the wave equation describing the vibrations of a stretched string. Hyperbolic partial differential equations are very essential in engineering and theoretical physics problems. One of the famous problems of their applicability in theoretical physics is the solution of the motion of a relativistic quantum mechanical particle in an electromagnetic field. The problems of vibrating rectangular or circular membrane are also very interesting especially when the membrane is subject to a restorative force.

The vibrating membrane problem can be used as a rather appropriate example to demonstrate the power of computer algebra systems like Axiom, Maple, Mathematica, Derive, etc. [1]. Different methods have been applied for the investigation of vibrating membranes. The differential quadrature method was applied for frequency analysis of rectangular and circular membranes [2,3]. Accordingly, some important studies concerning analysis of membranes have been carried out [4,5]. Furthermore, free vibration analysis of plates and shells has been also investigated [6,7].

Once the boundary and initial conditions are given, the simplest method for solving the differential equation governing the problem of a vibrating rectangular or circular membrane is given, as usual [8], by separating the variables. In the presence of a restorative force, that is proportional to the velocity, the perturbation expansions for eigenvalues and eigenfunctions are also of particular interest. Based upon the known solutions of the problem in the absence of the restorative force one can then derive the solutions of the problem in the presence of the external force in the form of a power series of those solutions [9].

In the present paper, we have solved the differential equation, which represents the motion of a stretched elastic circular membrane that is subjected to a restorative force proportional to the velocity by using two methods. The first is the usual method of separation of variables and the second is the perturbation method. The roles of the restoring force and the initial displacement are discussed. Finally, the displacement of the membrane at any given point (r,θ) and instant of time t is given and the numerical solutions are then given by using the program Mathematica.

2. Formulation of the Problem

The vibrations of a circular membrane when the membrane is subjected to a restorative force proportional to the velocity at any instant of time t is governed by the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}+K\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right);u=u(r,\theta,t),\ K=\text{constant}. \quad (2.1)$$

subjected to the boundary conditions

$$u(a,\theta,t)=0, \quad (2.2)$$

and the initial conditions

$$u(r,\theta,0)=f(r,\theta), \quad (2.3)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r,\theta). \quad (2.4)$$

The initial displacement $\boldsymbol{f(r,\theta)}$ and initial velocity $\boldsymbol{g(r,\theta)}$ of the membrane are assumed to be continuous functions.

3. The Method of Separation of Variables

Let us find solutions of (2.1) subject to the boundary and initial conditions given by (2.2), (2.3) and (2.4) in the form

$$u=u(r,\theta,t)=T(t)R(r)\Theta(\theta)\,.$$

Hence, the partial differential equation (2.1) separates to three second-order ordinary differential equations in the form

$$\frac{d^2T}{dt^2}+k\frac{dT}{dt}=-c^2\lambda^2T \quad (3.1)$$

$$\frac{d^2\Theta}{d\theta^2}=-m^2\Theta, \quad (3.2)$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{m^2}{r^2}\right)R=O. \quad (3.3)$$

In equations (3.1), (3.2) and (3.3) m and λ are arbitrary constants to be determined. The solutions of these equations, which are finite in t and continuous in our domain of values of r and θ are simply given by

$$T(t)=e^{\frac{-kt}{2}}\{A_1cos(c\beta t)+B_1sin(c\beta t)\}, \quad (3.4)$$

$$\Theta(\theta) = A_2 cos(m\theta) + B_2 sin(m\theta), \tag{3.5}$$

$$R(r) = J_m(\lambda r), \tag{3.6}$$

Here, $m = 0, 1, 2, \dots\dots\dots$, $\beta = \sqrt{\lambda^2 - \frac{K^2}{4c^2}}$ and $J_m(\lambda r)$ are the Bessel functions. The boundary condition (2.2) implies that

$$R(a) = J_m(\lambda a) = 0, \tag{3.7}$$

Assuming that $j_{m,1}, j_{m,2}, j_{m,3}, \dots\dots\dots.$, are positive roots of $J_m(r)$, the solutions of (2.1) are then given by

$$u = e^{\frac{-kt}{2}} J_m\left(\frac{j_{m,n}}{a} r\right)\{A_{m,n} cos(m\theta) + B_{m,n} sin(m\theta)\}\{cos(c\beta_{m,n}t) + D_{m,n} sin(c\beta_{m,n}t)\}, \tag{3.8}$$

where $\beta_{m,n} = \sqrt{\frac{j_{m,n}^2}{a^2} - \frac{K^2}{4c^2}}$.

If the function $g(r,\theta)$ is taken to be zero, i.e. the initial velocity is zero, we obtain $D_{m,n} = \frac{K}{2c\beta_{m,n}}$, and the final solution in this case is then given by

$$u = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} e^{\frac{-kt}{2}} \left\{cos(c\beta_{m,n}t) + \frac{K}{2c\beta_{m,n}} sin(c\beta_{m,n}t)\right\} J_m\left(\frac{j_{m,n}}{a} r\right)\{A_{m,n} cos(m\theta) + B_{m,n} sin(m\theta)\} \tag{3.9}$$

4. The Perturbation Method of Solution

To apply the perturbation method [9,10] to equation (2.1) we try first to find solutions of the form

$$u = ve^{\frac{-kt}{2}}, \tag{4.1}$$

where v is also a function of r, θ and t.

Equation (2.1), then, becomes

$$\frac{\partial^2 v}{\partial t^2} - \frac{K^2}{4} v = c^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right), \tag{4.2}$$

with the boundary condition

$$v(a, \theta, t) = 0, \tag{4.3}$$

and the initial conditions

$$v(r, \theta, 0) = f(r, \theta), \tag{4.4}$$

$$v_{t=0} = g(r,\theta) + \frac{K}{2} f(r,\theta) = h(r,\theta), \tag{4.5}$$

The order of the approximation in the perturbation method is very important, so that we rewrite equation (4.2) in the form

$$c^2\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right)=v_{tt}-\alpha\frac{K^2}{4}v, \tag{4.6}$$

where α is a parameter introduced to know the order of the approximation and takes the value 1 in the final result. Accordingly, the zeroth order of the approximation is governed by the equation representing the case where there is no external restorative force, which is obtained by putting $\alpha=0$ in (4.6), i.e.

$$c^2\left(\frac{\partial^2 v^{(0)}}{\partial r^2}+\frac{1}{r}\frac{\partial v^{(0)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v^{(0)}}{\partial \theta^2}\right)=v_{tt}^{(0)}, \tag{4.7}$$

where $v^{(0)}$ is the corresponding solution in this case, which can be proved to be given by

$$v^{(0)}(r,\theta,t)=\sum_{m=o}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{j_{m,n}}{a}r\right)\{A_{m,n}cos(m\theta)+\ B_{m,n}sin(m\theta)\} \times\{cos(cj_{m,n}t)+D_{m,n}sin(cj_{m,n}t)\}. \tag{4.8}$$

The boundary condition (4.3) is then satisfied and the initial conditions are the same as given by (4.4) and (4.5), with $K=0$.

We have now to determine by how much the solutions of the vibrations of the circular membrane, given by equation (4.8), under the boundary and initial conditions stated above, have been changed on account of the presence of the disturbing factor $K\frac{\partial u}{\partial t}$, since it is assumed to be small compared to the other terms. The change is known as a perturbation.

The second step is now to use the solution $v^{(0)}(r,\theta,t)$, equation (4.8), to derive solutions of equation (4.6), satisfying the boundary and initial conditions stated above in the following manner. Assume that the solutions of equations (4.6), $v(r,\theta,t)$, are expanded in series in powers of α, such that

$$v=v^{(0)}+\alpha v^{(1)}+\alpha^2 v^{(2)}+\cdots\ . \tag{4.9}$$

For any order of the approximation, we have the system of inhomogeneous wave equations

$$c^2\left(\frac{\partial^2 v^{(j)}}{\partial r^2}+\frac{1}{r}\frac{\partial v^{(j)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v^{(j)}}{\partial \theta^2}\right)=v_{tt}^{(j)}-\frac{K^2}{4}v^{(j-1)}, \qquad j=1,2,3,\dots, \tag{4.10}$$

with the initial conditions

$$v^{(j)}(r,\theta,0)=0\ ,\quad v_{t=0}^{(j)}=0\ ,\ j=1,2,3,\dots\ . \tag{4.11}$$

The solutions of equations (4.10) are given by

$$v^{(j)}(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}W_{m,n}^{(j)}(t)J_m\left(\frac{j_{m,n}}{a}r\right)\{A_{m,n}cos(m\theta)+B_{m,n}sin(m\theta)\}, \tag{4.12}$$
$j=0,1,2,\dots,$

where

$$\mathcal{W}_{m,n}^{(0)}(t) = cos(cj_{m,n}t) + D_{m,n}sin(cj_{m,n}t), \tag{4.13}$$

and $\mathcal{W}_{m,n}^{(j)}(t)$, $j = 1, 2, 3, \ldots$, are to be determined. From (4.10) and (4.12) we get

$$\frac{d^2\mathcal{W}_{m,n}^{(j)}(t)}{dt^2} + (cj_{m,n})^2\, \mathcal{W}_{m,n}^{(j)}(t) = \frac{K^2}{4}\, \mathcal{W}_{m,n}^{(j-1)}(t),\ j = 1,2,3, \ldots \ldots \tag{4.14}$$

The solutions of (4.14) can be written in the form

$$\mathcal{W}_{m,n}^{(j)}(t) = \frac{K^2}{4cj_{m,n}}\int_0^t \mathcal{W}_{m,n}^{(j-1)}(\tau)sin\{cj_{m,n}(t-\tau)\}d\tau\ , = 1, 2, 3, \ldots . \tag{4.15}$$

Hence, the solutions of (2.1) are finally given by

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} u_{m,n}(r,\theta,t)$$

$$= e^{\frac{-Kt}{2}}\sum_{j=0}^{\infty}\alpha^j\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\mathcal{W}_{m,n}^{(j)}(t)J_m\left(\frac{j_{m,n}}{a}r\right)\{A_{m,n}\cos(m\theta) + +B_{m,n}sin(m\theta)\}. \tag{4.16}$$

From (4.14) and (4.15), we have

$$\mathcal{W}_{m,n}^{(1)}(t) = \frac{K^2}{8cj_{m,n}}\left[E_{m,n}t\ sin(cj_{m,n}t) + L_{m,n}\left(\frac{sin(cj_{m,n}t)}{cj_{m,n}} - t\ cos(cj_{m,n}t)\right)\right], \tag{4.17}$$

and

$$\mathcal{W}_{m,n}^{(2)}(t) = \frac{K^4}{128(cj_{m,n})^2}\left[M_{m,n}\left\{\frac{t\ sin(cj_{m,n}t)}{cj_{m,n}} - t^2cos(cj_{m,n}t)\right\} + N_{m,n}\left\{\frac{3}{(cj_{m,n})^2}sin(cj_{m,n}t) - \frac{3t}{cj_{m,n}}cos(cj_{m,n}t) - t^2sin(cj_{m,n}t) + \frac{3}{(cj_{m,n})^2}\{sin(cj_{m,n}t)\}^3 - \frac{6sin^2(cj_{m,n}t)cos(cj_{m,n}t)}{cj_{m,n}}\right\}\right]. \tag{4.18}$$

Accordingly, $v^{(1)}(r,\theta,t)$ and $v^{(2)}(r,\theta,t)$ are calculated. From which the functions v and, hence, u are calculated. We have applied the perturbation method to the second-order of the approximation, which is sufficiently enough since the convergence of the solutions for our choice of parameters is good. Hence, our solutions are finally given by

$$u(r,\theta,t) = e^{\frac{-Kt}{2}}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{j_{m,n}r}{a}\right)[A_{m,n}cos(m\theta) + B_{m,n}sin(m\theta)]\mathcal{W}_{m,n}(t)\ , \tag{4.19}$$

where

$$\mathcal{W}_{m,n}(t) = \{cos(cj_{m,n}t) + D_{m,n}sin(cj_{m,n}t)\} + \frac{K^2}{8cj_{m,n}}\left\{E_{m,n}t\ sin(cj_{m,n}t) + L_{m,n}\left(\frac{sin(cj_{m,n}t)}{cj_{m,n}} - t\ cos(cj_{m,n}t)\right)\right\} + \frac{K^4}{128(cj_{m,n})^2}\left[M_{m,n}\left\{\frac{t\ sin(cj_{m,n}t)}{cj_{m,n}} - t^2cos(cj_{m,n}t)\right\} + \right.$$

$$N_{m,n}\left\{\frac{3}{(cj_{m,n})^2}sin(cj_{m,n}t)-\frac{3t}{cj_{m,n}}cos(cj_{m,n}t)-t^2sin(cj_{m,n}t)+\frac{3}{(cj_{m,n})^2}\{sin(cj_{m,n}t)\}^3-\frac{6sin^2(cj_{m,n}t)cos(cj_{m,n}t)}{cj_{m,n}}\right\}\Big] \tag{4.20}$$

The initial condition (2.3) now gives

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{j_{m,n}r}{a}\right)\{A_{m,n}cos(m\theta)+B_{m,n}sin(m\theta)\}=f(r,\theta)\ , \tag{4.21}$$

and the initial condition (2.4) gives

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{j_{m,n}r}{a}\right)\{A_{m,n}cos(m\theta)+B_{m,n}sin(m\theta)\}\left(\frac{-K}{2}+c\lambda_{m,n}D_{m,n}\right)=g(r,\theta), \tag{4.22}$$

5. Determination of the Coefficients

In the treatment of the present problem we considered the following physical assumptions:

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the plane.
3. The tension per unit length, T, caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
4. The deflection $u(r,\theta,t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Although these assumptions cannot be realized exactly, they hold relatively accurately for small transverse vibrations of a thin elastic membrane, so that we shall obtain a good model, for instance, of a drumhead. In the numerical calculations we take $a=1$ ft, the density, $\rho=2$ slugs/ft², as for light rubber, the constant tension $T=8$ lb/ft, so that $c^2=T/\rho=4$(ft/sec)². Moreover, the initial displacement is taken to be

$$f(r,\theta)=\sum_{m=0}^{\infty}2^{-m}J_m(j_{m,1}r)cos(m\theta), \tag{5.1}$$

which is a continuous function of r and θ in the intervals $0\le r\le 1$ and $0\le\theta\le 2\pi$.

Also, as before, we assume that the initial velocity is zero, so that $g(r,\theta)=0$. Hence,

$$D_{m,n}=\frac{K}{2cj_{m,n}}, \tag{5.2}$$

The initial displacement condition then gives

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(j_{m,n}r\right)\{A_{m,n}cos(m\theta)+B_{m,n}sin(m\theta)\}=\sum_{m=0}^{\infty}2^{-m}J_m(j_{m,1}r)cos(m\theta). \tag{5.3}$$

Comparing the coefficients of $cos(m\theta)$ and $sin(m\theta)$ on both sides of (5.3) we conclude that all the coefficients are zero except:

$$A_{m,1}=2^{-m}\ , m=o,1,2,3,\dots\dots.... \tag{5.4}$$

Accordingly, (4.19) takes the form

$$u(r,\theta,t) = e^{\frac{-Kt}{2}} \sum_{m=0}^{\infty} J_m\left(j_{m,1} r\right) 2^{-m} \cos(m\theta) \mathcal{W}_{m,1}(t)\,, \tag{5.5}$$

where $\mathcal{W}_{m,1}(t)$ are given by (4.20) and (5.2).

It is to be noticed that the boundary and initial conditions for our model do not give information about the coefficients $E_{m,1}$, $L_{m,1}$, $M_{m,1}$ and $N_{m,1}$. We can arbitrarily take $L_{m,1} = 0$ and $N_{m,1} = 0$. Also, choosing $E_{m,1} = 1$ and $M_{m,1} = 1$ we get

$$W_{m,1}(t) = \cos\left(cj_{m,1}t\right) + \frac{K}{2cj_{m,1}} \sin\left(cj_{m,1}t\right) + \frac{K^2}{8cj_{m,1}}\left\{t \sin\left(cj_{m,1}t\right)\right\} + + \frac{K^4}{128\left(cj_{m,1}\right)^2}\left[\frac{t \sin\left(cj_{m,n}t\right)}{cj_{m,n}} - t^2 \cos\left(cj_{m,n}t\right)\right] \tag{5.6}$$

6. Results and Conclusions

In order to investigate the role of the restorative force, given in this paper by the term $K\frac{\partial u}{\partial t}$, we draw the maximum value of the function u as function of the time t for different values of the coefficient K. In Fig. (1) we present the variation of the maximum value of u with respect to the time t when there is no external force, $K = 0$. Fig.(2) shows the variations of maximum u with respect to t for $K = 0.1, 0.2, 0.3, 0.4$ and 0.5. It is seen from Fig.(2) that the amplitude of the oscillation and the time during which the phenomena is seen, decrease as K increases, as expected. The process is well presented at $K = 0.3$.

In Fig. (3) we present the vibrations of the circular membrane at $t = 0$, $K = 0.3$, initial displacement. In Figs. (4-21 "a") we present the vibrations of the circular membrane, by using the method of separation of variables, at $t = 0.25, 0.5, 1, 1.5, 2, 3, , \ldots\ldots, 16, 27, 28$, respectively, $K = 0.3$. In Figs. (4-21 "b") we present the vibrations of the circular membrane, by using the perturbation method, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \ldots\ldots, 16, 27, 28$, respectively, $K = 0.3$.

It is seen from Figs. (4-21 "a, b") that the two methods of solutions give the same results (same vibrations), a result which shows that the perturbation calculations up to the second-order of the approximation give accurate solution to the original problem.

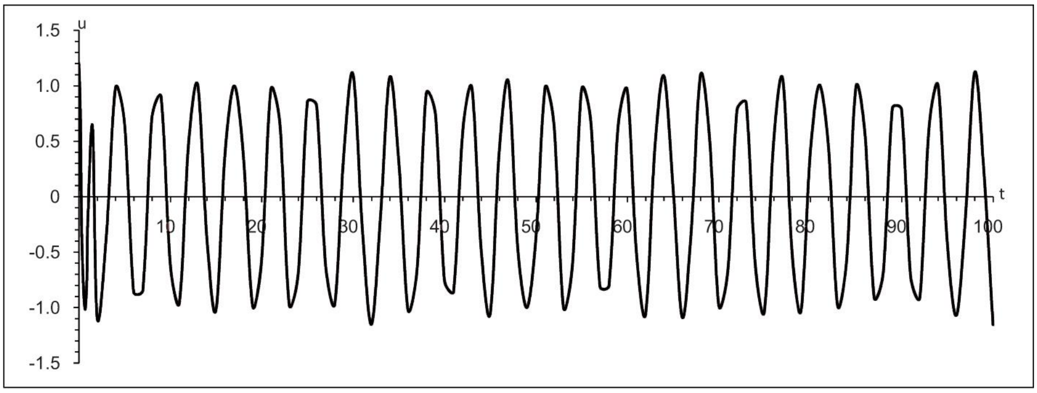


Fig. (1) Maximum displacement, for different values of r, θ, as function of time t, K = 0.0

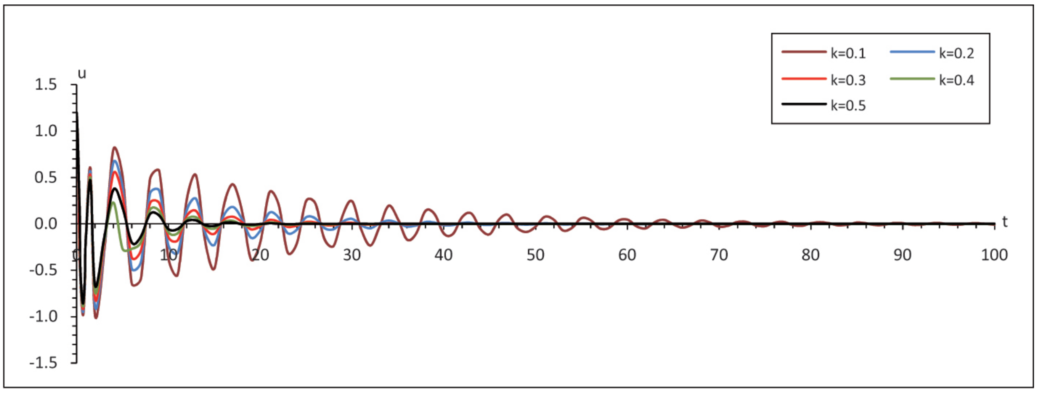


Fig. (2) Maximum displacement, for different values of r, θ, as function of time t, K = 0.1, 0.2, 0.3, 0.4 and 0.5.

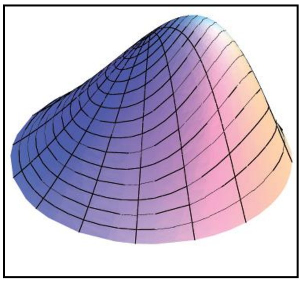
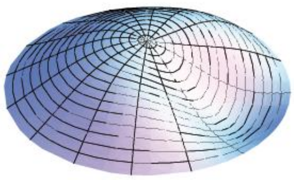
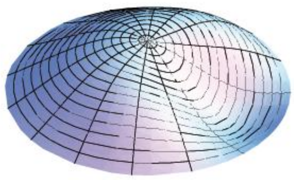
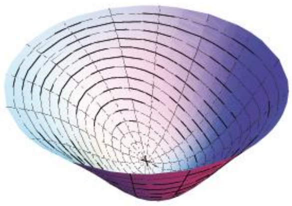
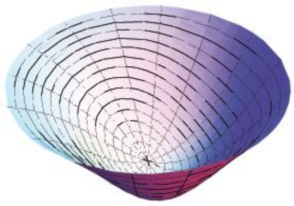
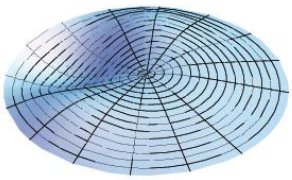
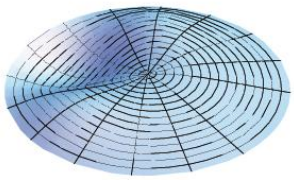
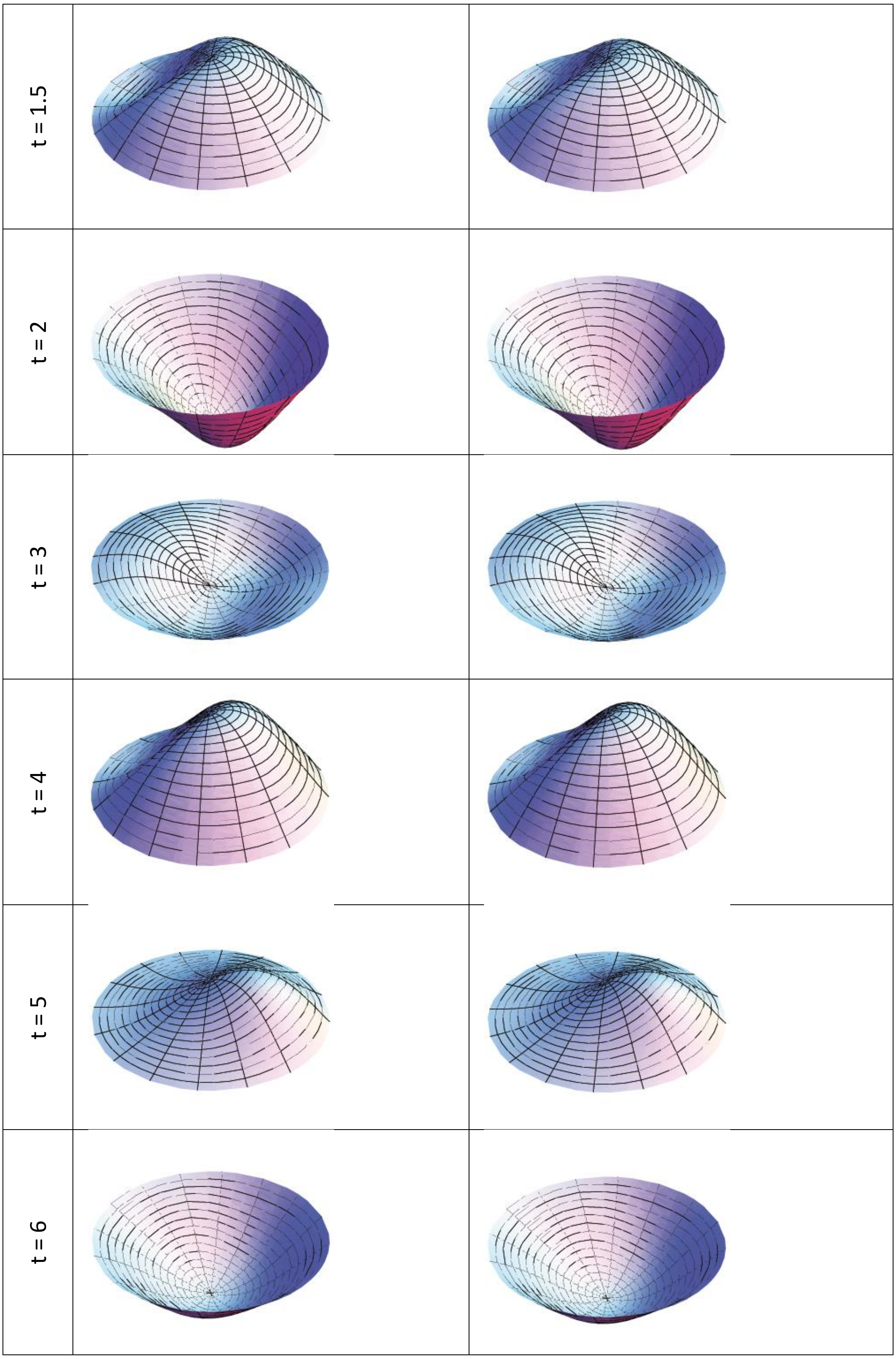
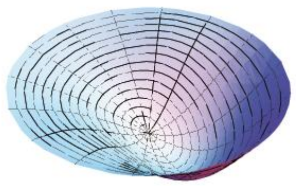
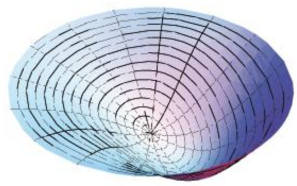
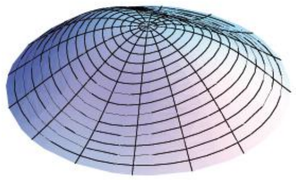
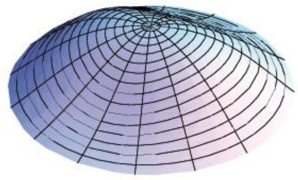
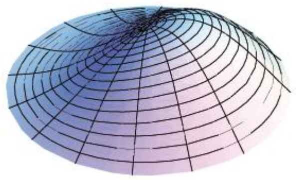
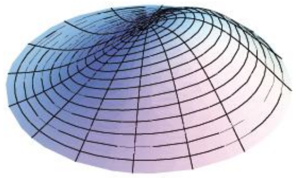
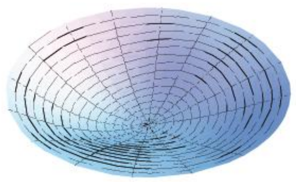
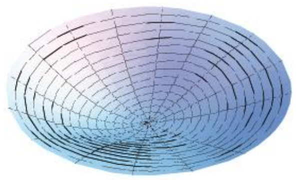
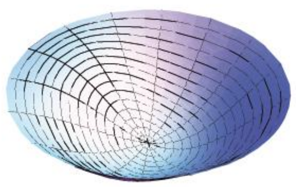
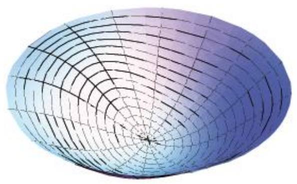
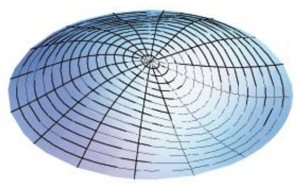
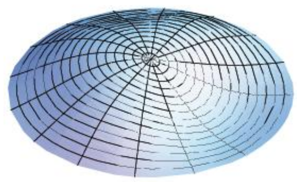
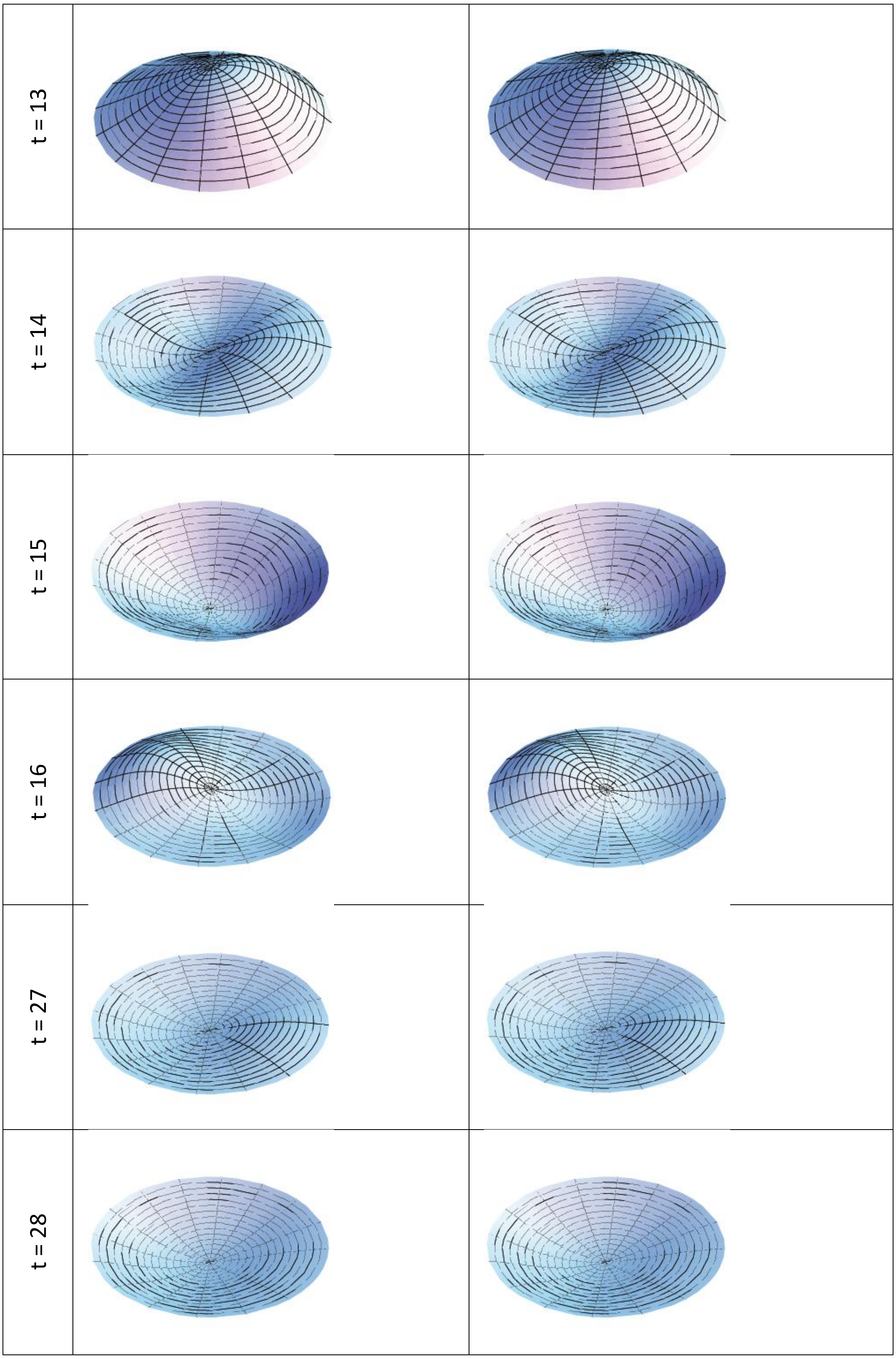


Fig. (3) Initial displacement.

Sec.	a- Method of separation of variables	b- Second-order perturbation method
t = 0.25		
t = 0.50		
t = 1.0		


t = 1.5
t = 2
t = 3
t = 4
t = 5
t = 6

t = 7		
t = 8		
t = 9		
t = 10		
t = 11		
t = 12		



Figs. (4-21 "a") Vibrations of the circular membrane, by using the method of separation of variables, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \ldots\ldots, 16, 27, 28$, respectively, $K = 0.3$.
Figs. (4-21 "b") Vibrations of the circular membrane, by using the perturbation method, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \ldots\ldots, 16, 27, 28$, respectively, $K = 0.3$.

Since the radius of the membrane $a = 1$ ft, the values of the frequency of oscillation, in our model, are given by

$$f_{m,1} = \frac{\lambda_{m,1} c}{2\pi} \text{HZ}. \quad (6.1)$$

The values of these frequencies are given in Table-1.

Table-1 Frequency values for $c = 2$ ft/sec.

	Frequency (HZ)
f_{01}	0.765
f_{11}	1.219
f_{21}	1.634
f_{31}	2.030
f_{41}	2.414
f_{51}	2.791
f_{61}	3.161
f_{71}	3.527
f_{81}	3.890
`f_{91}	4.250

In practical applications, the frequencies given by (6.1) produce vibrations which are outside the range of hearing. In circumstances like what we have studied, it is desirable from the physical point of view to take the value of the velocity as $c = 800$ ft/sec. In such case, the corresponding values of the frequency are in the range $306.052 \leq f_{m1} \leq 1699.523\ HZ, m = 0,1, \dots, 9$. For the considered modes, these frequencies produce vibrations which can be heard. At the same time, the shape of the resulting figures are very similar to those obtained in Figures (4-21 “a” and “b”) but with different amplitudes.

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