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On Some Integral Equations with Carleman Kernel

I. H. Elsirafy[1), S. B. Doma[2) and M. A. Elsayed[3)

1) Faculty of Science, Alexandria University, Alexandria, Egypt, Email: isirafy@yahoo.com
2) Faculty of Information Technology and Computer Sciences, Sinai University, El-Arish, North Sinai, Egypt; E-mail address: sbdoma@sci.alex.edu.eg
Permanent address: Faculty of Science, Alexandria University, Alexandria, Egypt.
3High Institute of Engineering, El-Shrouk Academy, Cairo, Egypt.

Abstract
In this paper, the existence and uniqueness of the solution of Volterra-Fredholm integral equations of the first kind with Carleman kernel is investigated. Furthermore, toeplitz matrix method and Nystrom method are used to obtain the eigenvalues and eigenfunctions for the Fredholm integral equation of the second kind with Carleman kernel. The comparison between the two methods shows that their numerical results are approximately the same.

Keywords
Volterra-Fredholm integral equation, Carleman kernel, eigenvalues and eigenfunctions, toeplitz matrix method, poduct Nystrom method.

1-Introduction

Some basic equations of mathematical physics and contact problems, in the theory of elasticity, lead to an integral equation of the first or second kind that requires solutions [1]. Accordingly, different methods of solving some Fredholm integral equations (FIE) of the first kind with Carleman function are discussed in [2,3].

The theory of eigenvalues and eigenfunctions are also playing an important role in solving the integral equations, especially with singular kernel. For example, the spectral relationships for an integral equation of Volterra- Fredholm integral equation (V-FIE) of the first kind can be obtained [3,4].

In the present paper, we study the existence and uniqueness of the solution of V-FIE of the first kind with Carleman kernel. Two numerical methods are used to obtain the eigenvalues and the eigenfunctions for the Fredholm integral equation of the second kind with Carleman kernel, namely: the Toeplitz matrix method and the Nystrom method. The comparison between the two methods shows that their numerical results are approximately the same.

2-Volterra-Fredholm Integral Equations of the First Kind with Carleman Kernel

2-1 Formulation of the problem

Let us consider the V-FIE of the first kind, namely

$$\int_0^t G(t,\tau)\Phi(x,\tau)d\tau - \int_0^t \int_{-1}^1 F(t,\tau)|x-y|^{-\nu}\Phi(y,\tau)dyd\tau = f(x,t), \quad (0<\nu<1) \qquad (2.1)$$

under the condition

$$\int_{-1}^{1} \Phi(x,t)dx = P(t) \tag{2.2}$$

The contact problem for a rigid surface (G, υ) having an elastic material lead to the integral equation (2.1) under the condition (2.2), which can be investigated for G being the displacement magnitude and ν the Poisson's coefficient. Let a stamp of length two units with its surface being described by $f_*(x)$ is impressed into an elastic layer surface of a strip by a variable force $P(t)$, whose eccentricity of application is $e(t)$, that causes rigid displacement $\gamma(t)$. Therefore, we define the free term of (2.1) as

$$f(x,t) = \pi\theta[\delta(t) - f_*(x)], \ \left(\theta = \frac{G}{2(1-\nu)}, 0 \le t \le \infty\right). \tag{2.3}$$

In (2.1) the given function of time $F(t,\tau)$ represents the resistance force of the lower material, while $G(t,\tau)$ is called the supplied external force in the contact domain of the upper and lower surfaces. Then, using the method of potential theory [5] the spectral relationships for the Gegenbauer operator are obtained and many special cases are discussed. Also, a numerical method is used to obtain a system of FIE of the first kind or second kind depending on the relation between the derivatives of the two functions $F(t,\tau)$ and $G(t,\tau)$ for all values of $t,\tau \in [0,T]$. Finally, we used the Toeplitz matrix method and Nystrom product method to obtain numerical solutions of the linear system of FIE with Carleman kernel.

2-2 Existence and Uniqueness of the Solution

In order to guarantee the existence of a unique solution of equation (2.1), under the condition (2.2), we assume the following:

(i) The kernel $k\left(\left|\frac{x-y}{\lambda}\right|\right)$ satisfies the discontinuity condition

$$\int_{-1}^{1}\int_{-1}^{1} k^2\left(\left|\frac{x-y}{\lambda}\right|\right) dxdy = A \qquad \text{(A is a constant)}$$

(ii) For all values of $t,\tau \in [0,T]$ the two continuous functions of time $F(t,\tau)$ and $G(t,\tau)$ satisfy $|\,F(t,\tau)| < B$, $|\,\mathrm{G}(t,\tau)| < C$.

(iii) The known function $f(x,y) \in L_2[-1,1] \times C[0,T]$, and its norm is defined as

$$\|f(x,y)\|_{L_2\times C} = \max_{0\le t\le T} \int_0^t \left\{\int_{-1}^{1} f^2(x,\tau)dx\right\}^{\frac{1}{2}} d\tau.$$

(iv) The unknown function $\Phi(\mathrm{x},\mathrm{t})$ behaves like $f(x,t)$ and satisfies Lipschitz condition with respect to the first argument and Holder condition for the second argument.

To obtain the solution of (2.1), under (2.2), we divide the interval $[0,T]$, as $0 = t_0 < t_1 < \cdots < t_N = T$ where, $t = t_j, j = 0,1,2,\ldots,N$, to get

$$\int_0^{t_j} G(t_j,\tau)\Phi(x,\tau)d\tau - \int_0^{t_j}\int_{-1}^{1} F(t_j,\tau)|x-y|^{-\nu}\Phi(y,\tau)dyd\tau = f(x,t_j), \tag{2.4}$$

under the condition

$$\int_{-1}^{1} \Phi(x,t_j)dx = P(t_j). \tag{2.5}$$

Hence, we have

$$\sum_{i=0}^{j} \nu_i \, G(t_j,t_i)\Phi(x,t_i) - \sum_{l=0}^{j} u_l F(t_j,t_l) \int_{-1}^{1} |x-y|^{-\nu}\Phi(y,t_l)dy + O(\hbar_j^{p}) + o(\hbar_j^{\tilde{p}}) = f(x,t_j), (\hbar_j = \max_{0\le i;l} h_j ; h_i = t_{i+1} - t_i) \quad (2.6)$$

where, $O(\hbar_j^{p})$ is the estimate error deduced from the approximate integral of the function $G(t,\tau)$ and $o(\hbar_j^{\tilde{p}})$ depends on $F(t,\tau)$. The values of the weight functions ν_i, u_l and $p, \tilde{p}$ depend on the number of derivatives of $G(t,\tau)$ and $F(t,\tau)$, for all $\tau \in [0,T]$ with respect to t.

Example

If $G(t,\tau) \in C^4[0,T]$
then, we have
$p = 4,\ j \approx 4$ and $\nu_0 = \frac{1}{2}h_0,\ \nu_4 = \frac{1}{2}h_4,\ \nu_n = h_n, n = 1,2,3,\ \nu_n = 0$ for $n > 4$.
While, if $F(t,\tau) \in C^3[0,T]$, we have
$\tilde{p} = 3, \tilde{k} \approx 3, u_0 = \frac{1}{2}h_0, u_3 = \frac{1}{2}h_3, u_m = h_m, m = 1,2$ and, $u_m = 0$ for $m > 3$.

More information about the characteristic points and quadratic coefficient are found in [6,7].

Using the following notations

$$G(t_j,t_l) = G_{j,i} \qquad F(t_j,t_l) = F_{j,i}, \Phi(x,t_i) = \Phi_i(x)$$
$$f(x,t_i) = f_i(x), (i,j,l = 0,1,2,\dots N), \quad (2.7)$$

formula (2.6), after neglecting the error, becomes

$$\sum_{i=0}^{j} \nu_i \, \mathrm{G_{j,i}}\Phi_i(x) - \sum_{l=0}^{j} u_l F_{j,l} \int_{-1}^{1} |x-y|^{-\nu}\Phi_l(y)dy = f_j(x) \quad (2.8)$$

under the condition

$$\int_{-1}^{1} \Phi_j(x)dx = P_j \quad (P_j \text{ are constants } j = 0,1,2,\dots N) \quad (2.9)$$

Now, we can discuss the following:

(a) Formula (2.8) represents a linear system of FIE of the second kind, for all cases when the two functions $G(t,\tau)$, $F(t,\tau)$ have the same derivatives with respect to time $t \in [0,T]$. Hence, we have

$$\mu_j\Phi_j(x) - \mu_j' \int_{-1}^{1} |x-y|^{-\nu}\Phi_j(y)dy = g_j(x) \quad (2.10)$$

where

$$g_j(x) = f_j(x) - \sum_{i=0}^{j-1} u_i \, \mathrm{G_{j,i}}\Phi_i(x) + \sum_{i=0}^{j-1} u_i F_{j,i} \int_{-1}^{1} |x-y|^{-\nu}\Phi_i(y)dy,$$

$$\left(\mu_j = \frac{h_j}{2} G_{j,j}, \mu_j' = \frac{h_j}{2} F_{j,j}, G_{j,j} \neq 0, F_{j,j} \neq 0, u_i = \nu_i\right)$$

(b) When the function $G(t,\tau)$ has n derivatives with respect to t , $n < j$, therefore formula (2.8) takes the following forms

$$\sum_{i=0}^{n} u_i\{G_{n,i}\Phi_i(x) - F_{n,i}\int_{-1}^{1}|x-y|^{-\nu}\Phi_i(y)dy\} = f_n(x),\ (n < j\,,\, j = 0,1,\ldots,N\,) \qquad (2.11)$$

$$\sum_{i=0}^{n} u_i\{G_{n,i}\Phi_i(x) - F_{n,i}\int_{-1}^{1}|x-y|^{-\nu}\Phi_i(y)dy\} = f_n(x) - \sum_{i=0}^{n}\beta_i\left(u_i, G_{n,i}, F_{n,i}\right)\Phi_i(x),\ (n < j, j = 0,1,\ldots,N). \qquad (2.12)$$

Formula (2.11) represents a linear system of FIE of the second kind, while formula (2.8) is of the first kind. $\Phi_i(x)$, $i = 0,1,\ldots,n$ in the R.H.S. of (2.12) represents the recurrence solution of the integral equation (2.11) and b_i are constants.

(c) When the function $F(t,\tau)$ has n derivatives such that $n < j$, hence we have

$$\sum_{i=n+1}^{j} u_i G_{j,i}\Phi_i(x) = f_j(x) - \sum_{i=0}^{n}\gamma_i\left(u_i, G_{n,i}, F_{n,i}\right)\Phi_i(x), \qquad (2.13)$$

where $\Phi_i(x)$ in the R.H.S. is the solution of (2.11) and γ_i are constants.

3-Spectral Relationships for Carleman Integral Equation

In this section, using the method of potential theory, we obtain the spectral relationships for the FIE of the first kind with Carleman kernel. The importance of Carleman function came from the work of Arytonian [6], who has showed that the plane contact problem of the nonlinear theory of plasticity, in its first approximation can be reduced to FIE of the first kind with Carleman kernel.

Consider the integral equation

$$\int_{-1}^{1}|x-y|^{-\nu}\phi(y)dy\} = f(x) \quad (0 < \nu < 1), \qquad (3.1)$$

under the static condition

$$\int_{-1}^{1}\phi(y)dy\} = P \qquad (P \text{ is constant}) \qquad (3.2)$$

To solve (3.1), under the condition (3.2), we introduce the general Carleman function

$$U(x,t) = \int_{-1}^{1}\frac{\phi(y)}{[(x-y)^2+t^2]^{\frac{\nu}{2}}}dy. \qquad (3.3)$$

The solution of (3.3), under (3.2) is equivalent to the boundary value problem

$$\Delta U + \frac{\mu}{t}\frac{\partial U}{\partial t} = 0 \qquad \left((x,t) \notin (-1,1), \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right)$$
$$U(x,0) = f(x), \qquad (x,t \in (-1,1))$$
$$U(x,t) \cong P_r^{-\nu},\ \left(P \to 0\ as\ r = \sqrt{a^2+t^2} \to \infty\right) \qquad (3.4)$$

The complete solution of (3.3) is given by [4]

$$\phi(x) = \frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu+1}{2}\right)} \lim_{t\to 0} t|y|^{\nu} \frac{\partial U}{\partial t} \qquad x \in (-1,1) \tag{3.5}$$

where $\Gamma(n)$ is the Gamma function.

Using the substitution

$$U(x,t) = |t|^{-\frac{\nu}{2}} V(x,t), \tag{3.6}$$

and the transformation mapping

$$z = \frac{1}{2} w(\xi) = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right), \qquad \left(\xi = \rho e^{\iota\theta}, z = x + \iota y, \iota = \sqrt{-1}\right) \tag{3.7}$$

the boundary value problem (3.4), yields

$$\Delta V(\rho,\theta) + \nu(2-\nu)\left[\frac{1}{(\rho^2-1)^2} + \frac{1}{\rho^2 4\sin^2\theta}\right] V(\rho,\theta) = 0, \quad (\rho < 1)$$

$$\left[\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\theta\right]^{-\frac{\nu}{2}} V(\rho,\theta)|_{\rho=1} = f(\cos\theta), \quad (-\pi < \theta < \pi)$$

$$V(0,\theta) = 0, \qquad \left(\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{3.8}$$

where

$$V(x,y) = V\left(\left(\rho + \frac{1}{\rho}\right)\cos\theta, \left(\rho - \frac{1}{\rho}\right)\sin\theta\right) = V(\rho,\theta).$$

The transformation mapping (3.7) maps the region in the x-y plane into the region outside the unit circle γ, such that $\omega'(\xi)$ does not vanish or become infinite outside γ. The mapping function (3.7) maps the upper and lower half-plane $(x,y) \in (-1,1)$ into the lower and upper of semi-circle $\rho = 1$, respectively.

Moreover, the point $z = \infty$ will be mapped onto the point $\xi = 0$

Now, using the method of separation of variables, we can write

$$V(\rho,\theta) = R(\rho)Z(\theta) \tag{3.9}$$

The first differential equation of (3.8), then, becomes

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left[\nu(2-\nu)\frac{\rho^2}{(1-\rho^2)} - \alpha^2\right] R(\rho) = 0 \quad (0 \le \rho < 1) \tag{3.10}$$

and

$$\frac{d^2Z}{d\theta^2} + \rho^2\left[\alpha^2 + \frac{\nu(2-\nu)}{4\sin^2\theta}\right] Z(\theta) = 0, \qquad (-\pi < \theta < \pi) \tag{3.11}$$

where α^2 is the constant of separation.

The general solution of (3.10) and (3.11), respectively takes the form

$$R(\rho) = \rho^{n+\nu}(1-\rho^2)^{\nu} F\left(\frac{\nu}{2}, n+\nu; n+1; \rho^2\right), (R(0) = 0, 0 \le \rho < 1, n = 0,1,2,\dots) \tag{3.12}$$

and

$$Z(\theta) = |\sin\theta|^{\frac{\nu}{2}} C_n^{\frac{\nu}{2}}(\cos\theta). \qquad (-\pi < \theta \le \pi, n = 0,1,2,\dots) \qquad (3.13)$$

Here, $F(a,b;c;z)$ is Hypergeometric function and $C_n^{\frac{\nu}{2}}(x)$ is the Gegenbauer polynomial.
From (3.12), (3.13), (3.9), and using the result of (3.6), we get

$$\mathrm{U}(\rho,\theta) = \rho^{n+\nu}\, \mathrm{F}\left(\frac{\nu}{2}, n+\nu; n+\frac{\nu}{2}+1; \rho^2\right) C_n^{\frac{\nu}{2}}(\cos\theta)$$
$$\mathrm{U}(\rho,\theta) = U\left(\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\cos\theta, \frac{1}{2}\left(\rho-\frac{1}{\rho}\right)\sin\theta\right) = U(x,y) \qquad (3.14)$$

The complete solution of the problem, can be obtained, by writing (3.5) in polar coordinates

$$\phi(\cos\theta) = \frac{\Gamma\left(\frac{\nu}{2}\right)(\sin\theta)^{\nu-1}}{\sqrt{\pi}2^{\nu+1}\Gamma\left(\frac{\nu+1}{2}\right)} \lim_{\rho\to 1}(1-\rho^2)^{\nu}\frac{\partial U}{\partial \rho} \qquad (0 < \theta < \pi) \qquad (3.15)$$

Then, substituting from (3.14) into (3.15) we get

$$\phi(\cos\theta) = \frac{\Gamma(\nu)\Gamma\left(n+1+\frac{\nu}{2}\right)\sin\theta}{\sqrt{\pi}2^{\nu}\Gamma\left(\frac{\frac{\nu}{2}+1}{2}\right)\Gamma(n+\nu)} C_n^{\frac{\nu}{2}}(\cos\theta) \qquad (3.16)$$

Hence, inserting (3.16) in (3.1), we arrive at the following spectral relationships

$$\int_{-1}^{1} \frac{C_n^{\frac{\nu}{2}}(u)}{|x-u|^{\nu}(1-u^2)^{\frac{1-\nu}{2}}}du = \lambda_n C_n^{\frac{\nu}{2}}(x), \quad \lambda_n = \pi\,\Gamma(n+\nu)\left[n!\,\Gamma(\nu)\cos\left(\frac{\pi\nu}{2}\right)\right]^{-1}, \qquad (3.17)$$

where λ_n are called the eigenvalues of the integral operator. Many spectral relationships can be established from (3.17)

(a) Let $x = -1$ in (3.17) and use the following relation

$$C_n^{\frac{\nu}{2}}(-x) = (-1)^n C_n^{\frac{\nu}{2}}(x) \qquad (3.18)$$

we get

$$\int_{-1}^{1} \frac{C_n^{\frac{\nu}{2}}(1)}{|1+u|^{\nu}(1-u^2)^{\frac{1-\nu}{2}}}du = (-1)^n \lambda_n C_n^{\frac{\nu}{2}}(-1) \qquad (3.19)$$

(b) Differentiating (3.17) with respect to x and using the relation

$$\frac{d}{dx} C_n^{\nu}(x) = n C_{n-1}^{\nu+1}(x), \qquad (3.20)$$

we obtain

$$\int_{-1}^{1} \frac{C_n^{\frac{\nu}{2}}(1)}{|-1-u|^{\nu+1}(1-u^2)^{\frac{1-\nu}{2}}}du = \frac{-\pi}{(n-1)!\Gamma(1+\nu)\cos\left(\frac{\pi\nu}{2}\right)}\Gamma(n+\nu)\;\; C_{n-1}^{\nu+1}(-1) \qquad (3.21)$$

(c) Using the Gegenbauer $C_n^{\frac{\nu}{2}}(x)$ and the Jacobi $P_n^{(\alpha,\beta)}(x)$ relation

$$C_n^{\frac{\nu}{2}}(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma(n+\nu)}{\Gamma(\nu)\Gamma\left(n+\frac{\nu+1}{2}\right)} P_n^{\left(\frac{\nu-1}{2},\frac{\nu-1}{2}\right)}(x), \tag{3.22}$$

we obtain the following spectral relationships

$$\int_{-1}^{1} \frac{P_n^{\left(\frac{\nu-1}{2},\frac{\nu-1}{2}\right)}(u)}{|x-u|^{\nu}(1-u^2)^{\frac{1-\nu}{2}}} du = \lambda_n P_n^{\left(\frac{\nu-1}{2},\frac{\nu-1}{2}\right)}(x) \tag{3.23}$$

(d) Using the famous formulas [8]

$$\lim_{\nu\to 0} \Gamma\left(\frac{\nu}{2}\right) C_{2n}^{\frac{\nu}{2}}(x) = \frac{1}{n} T_{2n}(x),$$

and

$$\ln\frac{1}{|x-y|} = \lim_{\nu\to 0}(|x-y|^{-\nu}-1)^{-\nu}, \tag{3.24}$$

we arrive at the following spectral relationships

$$\int_{-1}^{1} \ln\frac{1}{|x-y|}\frac{T_{2m}(y)}{\sqrt{1-y^2}} dy = \begin{cases} \pi \ln 2 & m = 0 \\ \frac{\pi T_{2m}(x)}{2m} & m \geq 0 \end{cases} \tag{3.25}$$

where $T_{2m}(x)$ is the Chebyshev polynomial of the first kind.

For a Volterra-Fredholm integral operator, we have

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-1}^{1} \frac{C_{n_j}^{\frac{\nu}{2}}(u)}{|x-u|^{\nu}(1-u^2)^{\frac{1-\nu}{2}}} du = \sum_{j=0}^{k} u_j F_{j,k} \lambda_{n_j} C_{n_j}^{\frac{\nu}{2}}(x) \quad (n_j \geq 0) \tag{3.26}$$

4-Solution of Fredholm Integral Equation of the Second Kind

Let $j = 0$ in (2.10), we have

$$\Phi_0(x) - \lambda \int_{-1}^{1} |x-y|^{-\nu}\, \Phi_0(y) dy = f(x), \qquad \lambda = \frac{\mu_0'}{\mu_0}, \; f(x) = \frac{g(x)}{\mu_0} \tag{4.1}$$

In general, consider

$$\Phi(x) = f(x) + \lambda \int_{-1}^{1} k(|x-y|)\, \Phi(y) dy, \tag{4.2}$$

where $\Phi(x)$ is the unknown function, $f(t)$ is a given function and is called the free term. Hence, the convolution kernel has a singularity and λ is a known constant. Formula (4.1) can be written in the integral operator form

$$(1-K)\Phi = f, \tag{4.3}$$

where

$$K\Phi = \lambda \int_{-1}^{1} k(|x-y|)\,\Phi(y)dy \qquad (4.4)$$

4-1 The existence of a unique solution

In order to discuss the existence of a unique solution, we assume the following conditions:

(i) The kernel satisfies the Fredholm condition

$$\left(\int_{-1}^{1}\int_{-1}^{1} k^2(|x-y|)dydx\right)^{\frac{1}{2}} \le A_1, \qquad A_1 \text{ is a constant.}$$

(ii) The given function $f(x)$, with its first derivatives, is continuous in $L_2[-1,1]$, and its norm is defined as

$$\|f\| = \left(\int_{-1}^{1} f^2(x)dx\right)^{\frac{1}{2}} = A_2, \qquad A_2 \text{ is a constant.}$$

(iii) The unknown function $\Phi(x)$ behaves as the known function $f(x)$ in $L_2[-1,1]$.

Now, to prove the existence of the solution, we will use the successive approximation method (Picard method), for this we construct a sequence functions $\Phi_n(x)$ defined by

$$\Phi_n(x) = f(x) + \lambda \int_{-1}^{1} k(|x-y|)\,\Phi_{n-1}(y)\,dy \qquad \Phi_0(x) = f(x). \qquad (4.5)$$

For ease of manipulation, it is convenient to introduce

$$\psi_n(x) = \Phi_n(x) - \Phi_{n-1}(x),$$

$$\psi_n(x) = \lambda \int_{-1}^{1} k(|x-y|)\,[\Phi_{n-1}(y) - \Phi_{n-2}(y)]\,dy, \qquad n = 1,2\ldots$$

Then, we have

$$\psi_n(x) = \lambda \int_{-1}^{1} k(|x-y|)\;\psi_{n-1}(y)dy \qquad (4.6)$$

And, we can deduce that

$$\Phi_n(x) = \sum_{i=0}^{n} \psi_i(x) \qquad (4.7)$$

Using the properties of the norm, we obtain

$$\|\psi_n(x)\| = |\lambda| \left\| \int_{-1}^{1} k(|x-y|)\;\psi_{n-1}(y)dy \right\|$$

By induction, we get

$$\|\psi_n(x)\| \le A_2(A_1\lambda)^n \qquad (4.8)$$

This bounds makes the sequence ψ_n converges; so that when $n \to \infty$, we have

$$\Phi(x) = \lim_{n\to\infty} \Phi_n(x) = \sum_{i=0}^{\infty} \psi_i(x) \le \frac{A_2}{1-A_1\lambda} \qquad (4.9)$$

Also, it is easily to prove that the existed solution is unique.

4-2 The normality and continuity of the integral operator
The normality of integral operator

$$K\Phi = \lambda \int_{-1}^{1} k(|x-y|)\,\Phi(y)dy,$$

can be proved as follows

$$\|K\Phi\| = |\lambda| \left\| \left(\int_{-1}^{1} k^2(|x-y|)dy \right)^{\frac{1}{2}} \left(\int_{-1}^{1} \Phi^2(y)dy \right)^{\frac{1}{2}} \right\| \le |\lambda| A_1 \|\Phi\|$$

i.e.

$\|K\| \le |\lambda| A_1$

Also, for the continuity, we have

$\|K\Phi_1 - K\Phi_2\| = |\lambda| \left\| \left(\int_{-1}^{1} k^2(|x-y|)dy \right)^{\frac{1}{2}} \left(\int_{-1}^{1} |\Phi_1(y) - \Phi_2(y)|^2 dy \right)^{\frac{1}{2}} \right\|$,$\|K\Phi_1 -$
$K\Phi_2\| = |\lambda| A_1 \|\Phi_1 - \Phi_2\|$. (4.10)

Since from (4.9) $|\lambda|\ A_1 < 1$, then the integral operator is a contraction operator.

5-Numerical Methods
(i) The Toeplitz matrix method
In this section, we present the Toeplitz matrix method [9-11] to obtain the numerical solution for Fredholm integral equation of the second kind with singular kernel. The idea of this method is to obtain a system of $2N + 1$ linear algebraic equations, where $2N + 1$ is the number of the discrimination points used.
Let us consider the Fredholm integral equation of the second kind

$$\Phi(x) = f(x) + \lambda \int_{-a}^{a} k(|x-y|)\,\Phi(y)dy. \tag{5.1}$$

The integral term in (5.1) can be written in the form

$$\int_{-a}^{a} k(|x-y|)\,\Phi(y)dy = \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}-\mathbf{1}} \int_{nh}^{nh+h} k(|x-y|)\,\Phi(y)dy, \quad h = \frac{a}{N} \tag{5.2}$$

The second step is to approximate the integral in the right hand side of (5.2) by

$$\int_{nh}^{nh+h} k(|x-y|)\,\Phi(y)dy = A_n(x)\phi(nh) + B_n(x)\phi(nh+h) + R, \tag{5.3}$$

where $A_n(x)$ and $B_n(x)$ are two arbitrary functions which will be determined and R is the estimate error. Putting $\phi(x) = 1, x$ in equation (5.3), we obtain a set of two equations in terms of two functions $A_n(x)$ and $B_n(x)$, where, in this case, we have $R = 0$. By solving these two equations, the functions $A_n(x)$ and $B_n(x)$ take the forms

$$A_n(x) = \frac{1}{h}\big((nh+h)I(x) - J(x)\big) \tag{5.4}$$

and

$$B_n(x) = \frac{1}{h}(J(x) - nhI(x)) \tag{5.5}$$

The values of $I(x)$ and $J(x)$ are

$$I(x) = \int_{nh}^{nh+h} k(|x-y|) \tag{5.6}$$

and

$$J(x) = \int_{nh}^{nh+h} yk(|x-y|)\,dy \tag{5.7}$$

Hence, the relation (5.2), becomes

$$\int_{-a}^{a} k(|x-y|)\,\phi(y)dy = \sum_{n=-N}^{N} D_n(x)\ \phi(nh), \tag{5.8}$$

where

$$D_n(x) = \begin{cases} A_{-N}(x) & ; n = -N \\ A_n(x) + B_{n-1}(x) & ; -N < n < N \\ B_{N-1}(x) & ; n = N \end{cases} \tag{5.9}$$

Furthermore, the integral equation (5.1), then, becomes

$$\phi(x) - \lambda \sum_{n=-N}^{N} D_n(x)\ \phi(nh) = f(x). \tag{5.10}$$

Now, if we put $x = mh$ in (5.10), we get

$$\phi(mh) - \lambda \sum_{n=-N}^{N} a_{n,m}\ \phi(nh) = f(mh) \tag{5.11}$$

The function ϕ is a vector of $2N+1$ elements but $a_{m,n}$ is a matrix whose elements are given by

$$a_{n,m} = a'_{|n,m|} + g_{n,m} \tag{5.12}$$

$$a'_{|n,m|} = A_n(mh) + B_{n-1}(mh) \qquad ; -N \le n \le N$$

The matrix $a'_{m,n}$ is the Toeplitz matrix of order $2N+1$ where $-N \le m,n \le N$ and the elements of the second matrix are zeros except for the elements of the first and last rows. We can evaluate the values of the first row by substituting in $B_{n-1}(mh)$; by $-N; m = -N + i,\ 0 \le i \le 2N$, i is an integer. And the values of the last row are given by substituting in $A_n(mh)$; by $n = N, m = -N + i$.

The solution of formula (5.11) takes the form

$$\phi(mh) = \left[1 - \lambda a_{n,m}\right]^{-1} f(mh) \qquad \left|I - \lambda a_{n,m}\right| \neq 0, \tag{5.13}$$

where I is the unit matrix.

The Toeplitz matrix method is said to be convergent of order r in $[-a, a]$.
If, for N sufficiently large, there exists a constant $D > 0$ independent of N such that

$$\|\phi(x) - \phi_N(x)\| \leq DN^{-r} \tag{5.14}$$

The error term R is determined from the following formula

$$R = \left| \int_{nh}^{nh+h} y^2 k\,|x - y| dy - A_n(x)(nh)^2 - B_n(x)(nh + h)^2 \right| = O(h^3) \tag{5.15}$$

Application of Toeplitz matrix method

Consider the discontinuous kernel

$$k(x, y) = |x - y|^{-\nu} \qquad 0 \leq \nu < 1 \tag{5.16}$$

then

$$I(x) = \int_a^{a+h} |x - y|^{-\nu} dy = A_n(x) + B_n(x) \tag{5.17}$$

and

$$J(x) = \int_a^{a+h} y|x - y|^{-\nu} dy = aA_n(x) + (a + h)\, B_n(x) \tag{5.18}$$

Hence, using equations (5.17) and (5.18) we get

$$A_n(x) = \frac{1}{h}\left[\frac{-h}{1-\nu}(x - a)^{1-\nu} + \frac{|x-(a+h)|^{2-\nu}}{(1-\nu)(2-\nu)} - \frac{|x-a|^{2-\nu}}{(1-\nu)(2-\nu)}\right], \tag{5.19}$$

$$B_n(x) = \frac{1}{h}\left[\frac{h}{1-\nu}(x - (a + h))^{1-\nu} - \frac{(x-(a+h))^{2-\nu}}{(1-\nu)(2-\nu)} + \frac{(x-a)^{2-\nu}}{(1-\nu)(2-\nu)}\right]. \tag{5.20}$$

By putting $a = nh$, and $x = mh$, $-N \leq n \leq N$,$-N \leq m \leq N$ in equations (5.19) and (5.20), we get

$$A_n(mh) = \frac{h^{1-\nu}}{1-\nu}\left[(m - n)^{1-\nu}\left(-1 - \frac{m-n}{2-\nu}\right) + \frac{(m-n-1)^{2-\nu}}{2-\nu}\right], \tag{5.21}$$

and

$$B_n(mh) = \frac{h^{1-\nu}}{1-\nu}\left[(m - n - 1)^{1-\nu}\left(1 - \frac{m-n-1}{2-\nu}\right) + \frac{(m-n)^{2-\nu}}{2-\nu}\right]. \tag{5.22}$$

Therefore, the elements of the Toeplitz matrix are given by

$$a'_{n,m} = A_n(mh) + B_{n-1}(mh) = \frac{h^{1-\nu}}{(1-\nu)(2-\nu)}[(m - n - 1)^{2-\nu} - 2(m - n)^{2-\nu} + (m - n + 1)^{2-\nu}] . \tag{5.23}$$

In the homogeneous case, we have the following integral equations

$$K\phi = \lambda\phi$$

$$K\Phi = \int_{-1}^{1} |x - y|^{-\nu}\, \Phi(y) dy, \quad 0 < \nu < 1.$$

If we use the Toeplitz matrix method, the eigenvalues and eigenfunctions will be obtained numerically as follows, for $N = 1$, $\nu = 0.1$

$$a_{m,n} = \begin{pmatrix} .5847995322 & .526315790 & .4756686559 \\ 1.012942670 & 1.169590643 & 1.012942670 \\ .4756686559 & .526315790 & .584795322 \end{pmatrix}$$

and

Eigenvalues λ	Eigenfunctions	Average eigenfunctions
.08099027029	$[.3945573400, -.7342712425, .3945573552]$	.0182811509
.1091266661	$[.7409863181, -.110x10^{-7}, -.7409863062]$	$9x10^{-10}$
2.149064351	$[-.4352411919, -.9002270738, -.4352411920]$	-.5902364859

For $N = 2$, $\nu = 0.1$, we have

$$a_{n,m} = \begin{pmatrix} .3133840535 & .2820456481 & .2549045215 & .2430196326 & .2353518670 \\ .5428225358 & .6267681066 & .5428225364 & .5012146136 & .4806335778 \\ .5012146135 & .5428225364 & .6267681070 & .5428225364 & .5012146135 \\ .4806335778 & .5012146136 & .5428225364 & .6267681066 & .5428225358 \\ .2353518670 & .2430196326 & .2549045215 & .2820456481 & .3133840535 \end{pmatrix}$$

and

Eigenvalues λ	Eigenfunctions	Average eigenfunctions
.04614131009	[.3635923367,-.4691244764,.2494323361 ,-.4691244015,.3635922405]	7.676012832×10^{-3}
.04709772222	[.5330501467,-.4225289259,-.95x10^{-8} ,.4225289661,-.5330501731]	8.6×10^{-10}
.08445138050	[-.3147874395,-.1466550567,.8754431615 ,-.1466550617,-.3147874386]	-9.488367×10^{-3}
.1564879573	[.2390245966,.4805219940,-.18x10^{-8} ,-.4805219991,-.2390245984]	-1.74×10^{-9}
2.172894057	[.4323332395,.8958531334,.9093450733 ,.8958531327,.4323332389]	.7131435636

Finally, for $N = 3, \nu = 0.1$; we get

Eigenvalues λ	Eigenfunctions	Average eigenfunctions
.03189238343	[.4873130708,-.4432031295,.1354699398 ,-.1037x10^{-6}, -.1354697675 ,.4432026689,-.4873126619]	2.414285714×10^{-9}
.03242709156	[-.3821413186,.4378206084,-.1718382822 ,.1982746406,-.1718384881 ,.4378212561,-.3821420124]	$-4.863370886\times10^{-3}$
.04517354903	[.1970588112,-.03258804927,-.4296431679 ,.5513279056,-.4296431643 ,-.0325880455,.1970588071]	2.997585276×10^{-3}
.06123226648	[-.1517762035,-.1391825439,.3715950288 ,.9x10^{-9},-.3715950334 ,.1391825443,.1517762026]	-6×10^{-10}
.09567105961	[.2531157089,.4024654962,-.2839905569 ,-.6921617410,-.2839905427 ,.4024655015,.2531157020]	7.288509714×10^{-3}
.1662641344	[.2678353197,.6185749142,.4185740823 ,-.32x10^{-8},-.4185740846 ,-.6185749140,-.2678353144]	$-4.114285714\times10^{-9}$
2.178153958	[-.4322066284,-.8903750005,-.9085714372 ,-.9138674191,-.9085714375 ,-.8903750009,-.4322066284]	$-.8733799745$

(ii) The product Nystrom method

To use the product Nystrom method as a numerical method, we consider

$$\phi(x) = f(x) + \lambda \int_a^b p(x,y)\breve{k}(x,y)\,\phi(y)dy, \tag{5.24}$$

where $p(x,y)$ is 'badly behaved' function and $\breve{k}(x,y)$ is 'well behaved' function of their arguments, $f(x)$ is a given function, while $\phi(x)$ is the unknown function. Here, the use of product integration treats $p(x,y)$ exactly and approximates only the part of the integrand which is smooth, by a suitable Lagrange interpolation polynomial. So, equation (5.24) can be written in the form

$$\phi(x_i) = f(x_i) + \lambda \sum_{j=0}^{N} W_{ij}\breve{k}(x_i,y_j)\,\phi(y_j), \tag{5.25}$$

where, $x_i = y_i = a + ih, i = 0,1,2,\dots,N,$ with $h = \frac{b-a}{N}$, N even and W_{ij} are the weights which can be determined directly from [6,9].

Also, we approximate the integral term by a product integration, from Simpson's rule, where, $x = x_i$ and we write

$$\int_a^b p(x_i, y)\breve{k}\,(x_i, y)\,\phi(y)dy = \sum_{j=0}^{\frac{N-2}{2}} \int_{y_{2j}}^{y_{2j+2}} p(x_i, y)\breve{k}\,(x_i, y)\,\phi(y)dy. \qquad (5.26)$$

Hence, we get

$$\sum_{j=0}^{N} W_{ij}\breve{k}\,(x_i, y_j)\,\phi(y_j) = \sum_{j=0}^{\frac{N-2}{2}} \int_{y_{2j}}^{y_{2j+2}} p(x_i, y)\breve{k}\,(x_i, y)\,\phi(y)dy. \qquad (5.27)$$

If we approximate the nonsingular part of the integrand, $\breve{k}(x, y)\phi(y)$, by a second degree of Lagrange interpolation polynomial which interpolates it at the points $y_{2j}, y_{2j+1}, y_{2j+2}$, over the interval $[y_{2j}, y_{2j+2}]$, we obtain

$$\int_a^b p(y_i, y)\breve{k}\,(y_i, y)\,\phi(y)dy = \sum_{j=0}^{N} W_{ij}\breve{k}\,(y_i, y_j)\,\phi(y_j), \qquad (5.28)$$

where

$$W_{i,0} = \frac{1}{2h^2}\int_{y_0}^{y_2} p(y_i, y)(y_1 - y)(y_2 - y)dy,$$

$$W_{i2j+1} = \frac{1}{h^2}\int_{y_{2j}}^{y_{2j+2}} p(y_i, y)(y_{2j} - y)(y_{2j+2} - y)dy,$$

$$W_{i2j} = \int_{y_{2j}}^{y_{2j+2}} p(y_i, y)(y_{2j+1} - y)(y_{2j+2} - y)dy$$
$$+\frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y_{2j-2} - y)(y_{2j-1} - y)dy,$$

$$W_{iN} = \frac{1}{2h^2}\int_{y_{N-2}}^{y_N} p(y_i, y)(y_{N-2} - y)(y_{N-1} - y)dy$$

or, $W_{i0} = \beta_1(y_i)$, $W_{i2j+1} = 2\gamma_{j+1}(y_i)$ and $W_{i2j} = \alpha_j(y_i) + \beta_{j+1}(y_i)$, $W_{iN} = \alpha_{\frac{N}{2}}$ (5.29)

Therefore,

$$\alpha_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y_{2j-2} - y)(y_{2j-1} - y)dy$$

$$\beta_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y_{2j-1} - y)(y_{2j} - y)dy$$

$$\gamma_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y - y_{2j-2})(y_{2j} - y)dy. \qquad (5.30)$$

We, then, introduce the change of variables

$$y = y_{2j-2} + \zeta h \;, \qquad 0 \le \zeta \le 2.$$

Thus, the system (5.30) becomes

$$\alpha_j(y_i) = \frac{h}{2}\int_0^2 \zeta(\zeta-1)p\big(y_{2j-2}+\zeta h, y_i\big)d\zeta$$

$$\beta_j(y_i) = \frac{h}{2}\int_0^2 (\zeta-1)(\zeta-2)p\big(y_{2j-2}+\zeta h, y_i\big)d\zeta \qquad (5.31)$$

$$\gamma_j(y_i) = \frac{h}{2}\int_0^2 \zeta(2-\zeta)p\big(y_{2j-2}+\zeta h, y_i\big)d\zeta$$

If we, now, define

$$\psi_i = \int_0^2 \zeta^i p\big(y_{2j-2}+\zeta h, y_i\big)d\zeta \qquad i = 0,1,2 \qquad (5.32)$$

for $\quad p(x,y) = p(x-y),$

we get

$$\psi_i = \int_0^2 \zeta^i p\big(y_i - (\,y_{2j-2}+\zeta h)\big)d\zeta\;, \qquad i = 0,1,2$$
$$\big(y_i \; - y_{2j-2} = (i-2j+2)h\big).$$

Also, if we assume that $z = \; i-2j+2$, then we have

$$\alpha_j(y_i) = \frac{h}{2}\int_0^2 \zeta(\zeta-1)p((z-\zeta)h)d\zeta\; \beta_j(y_i) = \frac{h}{2}\int_0^2 (\zeta-1)(\zeta-2)p((z-\zeta)h)d\zeta, \qquad (5.33)$$
$$\gamma_j(y_i) = \frac{h}{2}\int_0^2 \zeta(2-\zeta)p((z-\zeta)h)d\zeta$$

Hence, the system (5.29) becomes

$$W_{i0} = \frac{h}{2}[2\psi_0(z) - 3\psi_1(z) + \psi_2(z)],\; z = i$$
$$W_{i2j+1} = h[2\psi_1(z) - \psi_2(z)],\; z = i-2j \qquad (5.34)$$
$$W_{i2j} = \frac{h}{2}[\psi_2(z) - \psi_1(z) + 2\psi_0(z-2) - 3\,\psi_1(z-2) + \;\psi_2(z-2)\,],\; z = i-2j+2$$
$$W_{iN} = \frac{h}{2}[\psi_2(z) - \psi_1(z)] \quad ,\; z = i-N+2$$

Therefore, the integral equation (5.24) is reduced to a system of linear algebraic equations of the form

$$(I-\lambda W)\Phi = F, \qquad (5.35)$$

which has the solution

$$\Phi = [I-\lambda W]^{-1}F, \qquad |I-\lambda W| \ne 0 \qquad (5.36)$$

The product Nystrom method is said to be convergent of order r in $[a,b]$ if and only if, for N sufficiently large there exists $C > 0$, independent of N, such that

$$\|\phi(x) - \phi_N(x)\|_\infty \leq CN^{-r}.$$

Application for Nystrom method
Here, for Carleman kernel if $k(x,y) = |x-y|^{-\nu}$, $0 < \nu < 1$,
then equation (5.24) takes the form

$$\phi(x) = f(x) + \lambda \int_a^b |x-y|^{-\nu}\, \phi(y)dy, \tag{5.37}$$

which can be written in the form

$$\phi(x) = f(x) + \lambda \sum_{j=0}^{N} W_{ij}\phi_0(y_j).$$

Using (5.30) we obtain

$$\alpha_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} |y_i - y|^{-\nu}\big(y_{2j-2} - y\big)\big(y_{2j-1} - y\big)dy$$
$$\beta_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} |y_i - y|^{-\nu}\big(y_{2j-1} - y\big)\big(y_{2j} - y\big)dy$$

and

$$\gamma_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} |y_i - y|^{-\nu}\big(y - y_{2j-2}\big)\big(y_{2j} - y\big)dy.$$

Also, letting

$$y = y_{2j-2} + \mu h, \quad 0 \leq \mu \leq 2$$

then

$$\alpha_j(y_i) = \frac{h^{1-\nu}}{2}\int_0^2 \mu(\mu-1)|i - 2j + 2 - \mu|^{-\nu}d\mu,$$
$$\beta_j(y_i) = \frac{h^{1-\nu}}{2}\int_0^2 (1-\mu)(2-\mu)|i - 2j + 2 - \mu|^{-\nu}d\mu,$$
$$\gamma_j(y_i) = \frac{h^{1-\nu}}{2}\int_0^2 \mu(2-\mu)|i - 2j + 2 - \mu|^{-\nu}d\mu.$$

Now, we define
$\Psi_i(x) = \int_0^2 \mu^i |x-\mu|^{-\nu}\, d\mu$, $i = 0,1,2$
This implies that

$$\alpha_j(y_i) = \frac{h^{1-\nu}}{2}[\Psi_2(x) - \Psi_1(x)],$$
$$\beta_j(y_i) = \frac{h^{1-\nu}}{2}[\Psi_0(x) - 3\Psi_1(x) + \Psi_2(x)],$$
$$\gamma_j(y_i) = \frac{h^{1-\nu}}{2}[\Psi_1(x) - \Psi_2(x)].$$

If we use Nystrom method, the eigenvalues and eigenfunctions will be obtained numerically as the follows

For $N = 2, \nu = 0.1$;

$$G = \begin{pmatrix} .2694026290 & 1.486642182 & .3621968690 \\ .3225806455 & 1.425641909 & .3225806455 \\ .3621968675 & 1.317079525 & .2694026295 \end{pmatrix}$$

where the eigenvalues and their corresponding eigenvectors are

Eigenvalues l	Eigenfunctions	Average eigenfunctions
$-.001938463057$	$[.4661736952, -.3643098490, 1.146079232]$	$.4159810261$
$-.09279423916$	$[.6010492989, -.424\times10^{-8}, -.6010492808]$	4.62×10^{-9}
2.059179873	$[.730032314, .7147503516, .6737142728]$	$.7061656461$

Secondly, by putting $N = 4, \nu = 0.1$; we get

$$G = \begin{pmatrix} .0191767844 & .2812229706 & .1523289351 & .3476285646 & .0762405237 \\ .0457657927 & .2507228340 & .1418998421 & .3351888646 & .0711432854 \\ .0655739037 & .1964416420 & .1291661815 & .3206970196 & .0655739044 \\ .0711432849 & .1244121432 & .0981931841 & .3032646016 & .0457657927 \\ .0762405269 & .1964416426 & .0383535691 & .2812229706 & .0191767847 \end{pmatrix}$$

Eigenvalues l	Eigenfunctions	Average eigenfunctions
$-.006483340777$	$[-.3479964908, -.7784662120, -5.333760501, .2062981233, 12.70483344]$	1.290181672
$-.01157210772$	$[.0045441955, -.2053567606, -.9106337374, -.0302181046, 2.712886059]$	$.3142443304$
$-.04129584962$	$[.3092173318, .1268631465, .06234040288, -.04093231118, -.6511369816]$	$-.03872968232$
$.04899001844$	[-.6337456416,-.6528335955,-.2790442080 ,.6431309750,-.2146679055]	$-.2274320751$
$.7318684659$	[-.746199751,-.7209673858,-.6614542798 ,-.5410163131,-.5276251099]	$-.6394525679$

And finally if we put $N = 6, \nu = 0.1$; we get

$$G = \begin{pmatrix} -.03226715575 & .0072752656 & .0114492660 & .0515456616 & .0295304828 & .0765665648 & .0109307545 \\ -.0145411503 & -.0130581588 & .0044965374 & .0432525284 & .0259250988 & .0711998784 & .0085710024 \\ -.0013357429 & -.0492456200 & -.0039925698 & .0335912984 & .0218321154 & .0653243848 & .0057753371 \\ .0023771779 & -.0972652860 & -.0246412348 & .0219696864 & .0170910597 & .0588259648 & .0023771783 \\ .0057753393 & -.0492456196 & -.06453431135 & .0072752656 & .0114492660 & .0515456616 & -.0013357425 \\ .0085710015 & -.0130581584 & -.0246412348 & -.0130581588 & .0044965374 & .0432525284 & -.0145411503 \\ .0109307639 & .0072752616 & -.0039925684 & -.0492456200 & -.0039925698 & .0335912984 & -.03226715560 \end{pmatrix}$$

We then have seven eigenvalues. We state just the real values and their corresponding eigenvectors which are

Eigenvalues l	Eigenfunctions	Average eigenfunctions
-.006910416283	[-.3596144275,-.0268741807,-.0737981465 ,-.8935235899,-1.986887244 ,.8338164976,3.001653859]	.070681824
-.01623664932	[.5404853146,.0824841180,-.1405357256 ,-.06928512220,-.6545016762 ,.2376930345,1.314909158]	.1873213002
.02840212825	[-.2231397056,-.2026229629,-.03365823290 ,.3043886336,-.0364652703 ,-.2776126958,-.4606684918]	−.1761004031

6-Comparison of Toeplitz Matrix Method and Product Nystrom Method

The following numerical results are obtained, when the exact solution is given by $\phi(x,t) = x + t$, and the Volterra kernel is $F(t,\tau) = t^2, G(t,\tau) = t, \alpha = -1$, $\beta = 0.8$. Also, we set in the Toeplitz matrix method the value of a equals 1 and in the product Nystrom method we set $a = -1, b = 1$. Here, ϕ'_{mn} means numerical method using Toeplitz matrix where $R^T$is the resulting error, while ϕ^N_{mn}, for the Nystrom method, and the resulting error is R^N. The dividing interval is considered when $h = 0.05$, $t = 0.3$, and $t = 0.8$.

Table 6.1: Results when $t = 0.3$

t	x	f_{mn}^{T}	R^{T}	f_{mn}^{N}	R^{N}
0.30	-1.00	-.700332E+00	.700332E-03	-.699997E+00	.327135E-05
0.30	-.95	-.649899E+00	.100601E-03	-.650020E+00	.199044E-04
0.30	-.90	-.600044E+00	.439576E-04	-.600034E+00	.340536E-04
0.30	-.85	-.550043E+00	.326429E-04	-.550052E+00	.518646E-04
0.30	-.80	-.500043E+00	.429652E-04	-.500068E+00	.677232E-04
0.30	-.75	-.450046E+00	.457629E-04	-.450083E+00	.834994E-04
0.30	-.70	-.400048E+00	.483623E-04	-.400099E+00	.965124E-04
0.30	-.65	-.350050E+00	.500296E-04	-.350113E+00	.113131E-03
0.30	-.60	-.300051E+00	.512533E-04	-.300127E+00	.127267E-03
0.30	-.55	-.250052E+00	.521324E-04	-.250241E+00	.141032E-03
0.30	-.50	-.200053E+00	.527657E-04	-.200154E+00	.154416E-03
0.30	-.45	-.150053E+00	.532102E-04	-.150167E+00	.167466E-03
0.30	-.40	-.100054E+00	.535064E-04	-.100180E+00	.180183E-03
0.30	-.35	-.500537E-01	.536825E-04	-.501926E-01	.192589E-03
0.30	-.30	-.537590E-04	.537590E-04	-.204686E-03	.204686E-03
0.30	-.25	.499462E-01	.537511E-04	-.497835E-01	.216482E-03
0.30	-.20	.999463E-01	.536700E-04	-.997720E-01	.227978E-03
0.30	-.15	.149946E+00	.535242E-04	.149761E+00	.239175E-03
0.30	-.10	.199947E+00	.533201E-04	.199750E+00	.250070E-03
0.30	-.05	.249947E+00	.530623E-04	.249739E+00	.260658E-03
0.30	.00	.299947E+00	.527544E-04	.299729E+00	.270932E-03
0.30	.05	.349948E+00	.523986E-04	.349719E+00	.280880E-03
0.30	.10	.399948E+00	.519963E-04	.399710E+00	.290494E-03
0.30	.15	.449948E+00	.515479E-04	.449700E+00	.299752E-03
0.30	.20	.449949E+00	.510532E-04	.499691E+00	.308642E-03
0.30	.25	.549949E+00	.505108E-04	.549683E+00	.317130E-03
0.30	.30	.599950E+00	.499187E-04	.599675E+00	.325199E-03
0.30	.35	.649951E+00	.492736E-04	.649667E+00	.332797E-03
0.30	.40	.699951E+00	.485711E-04	.699660E+00	.339899E-03
0.30	.45	.749952E+00	.478054E-04	.749654E+00	.346425E-03
0.30	.50	.799953E+00	.469682E-04	.799640E+00	.352333E-03
0.30	.55	.849954E+00	.460490E-04	.849643E+00	.357492E-03
0.30	.60	.899955E+00	.450328E-04	.899638E+00	.361830E-03
0.30	.65	.949956E+00	.438992E-04	.949635E+00	.365114E-03
0.30	.70	.999957E+00	.426182E-04	.999633E+00	.367200E-03
0.30	.75	.104996E+00	.411440E-04	.104963E+01	.367608E-03
0.30	.80	.109996E+01	.394013E-04	.109963E+01	.365972E-03
0.30	.85	.114996E+01	.372517E-04	.114964E+01	.361054E-03
0.30	.90	.119997E+01	.348866E-04	.119965E+01	.352328E-03
0.30	.95	.124997E+01	.29782E-04	.1249967E+01	.332114E-03
0.30	1.00	.129999E+01	.142894E-04	.129980E+01	.199793E-03

Table 6.2: Results when $t = 0.8$

t	x	f_{mn}^T	R^T	f_{mn}^N	R^N
0.80	-1.00	-.200841E+00	.841376E-03	-.199996E+00	.414027E-05
0.80	-.95	-.149659E+00	.341370E-03	-.150056E+00	.564264E-04
0.80	-.90	-.100104E+00	.104324E-03	-.100095E+00	.950709E-04
0.80	-.85	-.500568E+01	.567709E-04	-.501424E-01	.142431E-03
0.80	-.80	-.920406E-04	.920406E-04	-.184343E-03	.184343E-03
0.80	-.75	.499008E-01	.992013E-04	.497738E-01	.226157E-03
0.80	-.70	.998925E-01	.107486E-03	.997342E-01	.265765E-03
0.80	-.65	.149887E+00	.112607E-03	.149696E+00	.304337E-03
0.80	-.60	.199883E+00	.116528E-03	.199658E+00	.341540E-03
0.80	-.55	.249881E+00	.119409E-03	.249622E+00	.377734E-03
0.80	-.50	.299878E+00	.121572E-03	.299587E+00	.412864E-03
0.80	-.45	.349877E+00	.123176E-03	.349553E+00	.447081E-03
0.80	-.40	.399876E+00	.124344E-03	.399520E+00	.480374E-03
0.80	-.35	.449875E+00	.125159E-03	.449487E+00	.512820E-03
0.80	-.30	.499874E+00	.125681E-03	.499456E+00	.544413E-03
0.80	-.25	.549874E+00	.125957E-03	.549425E+00	.575190E-03
0.80	-.20	.599874E+00	.126020E-03	.599395E+00	.605144E-03
0.80	-.15	.649874E+00	.125895E-03	.649366E+00	.634284E-03
0.80	-.10	.699674E+00	.125603E-03	.699337E+00	.662601E-03
0.80	-.05	.749875E+00	.125159E-03	.749310E+00	.690082E-03
0.80	.00	.799875E+00	.124572E-03	.799283E+00	.716713E-03
0.80	.05	.849876E+00	.123852E-03	.849258E+00	.742459E-03
0.80	.10	.899877E+00	.123003E-03	.899233E+00	.767300E-03
0.80	.15	.949878E+00	.122027E-03	.949209E+00	.791175E-03
0.80	.20	.999879E+00	.120926E-03	.999186E+00	.814056E-03
0.80	.25	.104988E+01	.119696E-03	.104916E+01	.835847E-03
0.80	.30	.109988E+01	.118335E-03	.109914E+01	.856509E-03
0.80	.35	.114988E+01	.116834E-03	.114912E+01	.875895E-03
0.80	.40	.119988E+01	.115183E-03	.119911E+01	.893944E-03
0.80	.45	.124989E+01	.113368E-03	.124909E+01	.910432E-03
0.80	.50	.129989E+01	.111370E-03	.129970E+01	.925256E-03
0.80	.55	.134989E+01	.109162E-03	.134906E+01	.938053E-03
0.80	.60	.139989E+01	.106708E-03	.139905E+01	.948641E-03
0.80	.65	.144990E+01	.103957E-03	.144904E+01	.956375E-03
0.80	.70	.149990E+01	.100835E-03	.149904E+01	.960886E-03
0.80	.75	.154990E+01	.972298E-03	.154904E+01	.960849E-03
0.80	.80	.159991E+01	.929539E-04	.159904E+01	.955322E-03
0.80	.85	.164991E+01	.876645E-04	.164906E+01	.940902E-03
0.80	.90	.169992E+01	.805990E-04	.169908E+01	.916350E-03
0.80	.95	.174993E+01	.692591E-04	.174914E+01	.860546E-03
0.80	1.00	.179997E+01	.320161E-04	.179949E+01	.507466E-03

Conclusion

From Tables 6-1 and 6-2 we can see that the error obtained by using Toeplitz method is greater than the corresponding error obtained by using the product Nystrom method, for the first three values and after that we see that the error obtained by using the product Nystrom method is greater than the corresponding error obtained by using the Toeplitz method (for $t = 0.3, t = 0.8$).

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