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On the existence and stability of positive solution for a nonlinear fractional-order differential equation and some applications

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Abstract

We are concerned here with a class of nonlinear fractional-order differential equations. We study the existence of a unique positive solution, its uniform stability and its global stability at the equilibrium points. The fractional-order logistic equation, replicator (hawk-dove (HD) game) equation, law of mass actions and some other examples will be considered as applications.

Keywords: Fractional-order differential equations; Global stability; Lyapunov stability; Equilibrium points; Logistic equation; Replicator equation; Riccati's equation; Law of mass action.

1 Introduction

First of all we give the definition of fractional-order integration and fractional-order differentiation

Definition 1.1 *The fractional integral of order $\beta \in R^+$ of the function $f(t),\ t \geq a$ is defined by ([8], [9], [12] and [13])*

$$I_a^{\beta} f(t) \ = \ \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds. \tag{1.1}$$

The (Caputo) fractional derivative of order $\alpha \in (n-1, n)$ of $f(t),\ t \geq a$ is defined by

$$D_a^{\alpha}\ f(t) \ = \ I_a^{n-\alpha}\ D^n\ f(t), \quad D = \frac{d}{dt}. \tag{1.2}$$

Let $\alpha \in (0,1]$. The uniform stability of the solution of the initial value problems of the non-autonomous systems of fractional order

$$D_{t_0}^{\alpha}\, x(t) \;=\; A(t)\, x(t) \;+\; f(t), \quad t > t_o \quad and \quad x(t_o) \;=\; x_o$$

and

$$\frac{d}{dt}x(t) \;=\; A(t)\, \frac{d}{dt}\, I_{t_0}^{\alpha}\, x(t) \;+\; f(t), \quad t > t_o \quad \text{and} \quad x(t_o) \;=\; x_o$$

has been studied in [1].
The equilibrium points of the initial value problems of the logistic equation

$$D^{\alpha}x(t) = \rho\, x(t)(1 - x(t)), \quad t,\ \rho > 0, \quad \text{and} \quad x(0) = x_o, \tag{1.3}$$

and of the fractional-order replicator (hawk-dove (HD) game) equation

$$D^{\alpha}x(t) = \rho\, x(t)(1 - x(t))(A - Bx(t)), \quad A,\ B,\ t,\ \rho > 0, \quad \text{and} \quad x(0) = x_o \tag{1.4}$$

have been studied in [7] and [2] respectively. The authors in [7] and [2] evaluated the equilibrium points from the equation $D^{\alpha}x(t) = 0$ not from the equation $\frac{d}{dt}x(t) = 0$ as usual. Theorem 3 here proved that these results are true and Theorem 2 proved the global stability of the solutions of (1.3) and (1.4).

Now let $a_k(t),\ t \in I = [0,T],\ k = 0,1,2,\cdots$ are given functions. We are concerned here with the initial value problem of the nonlinear fractional-order differential equation

$$D^{\alpha}x(t) = \sum_{k=0}^{n} a_k(t)\ f_k(x(t)) \tag{1.5}$$

with the initial data

$$x(0) = x_o. \tag{1.6}$$

The initial value problem (1.5) - (1.6) is a general case of the initial value problem

$$D^{\alpha}x(t) = \sum_{k=0}^{n} a_k(t)\ x^k(t), \quad t > 0 \quad \text{and} \quad x(0) = x_o \tag{1.7}$$

which has many applications. For examples the initial value problem of the fractional-order logistic equation (1.3), replicator equation (1.4), Ricati's equation [11]

$$D^{\alpha}x(t) \;=\; a_o(t) \;+\; a_1(t)\, x(t) \;+\; a_2(t)\, x^2(t), \quad t > 0 \quad \text{and} \quad x(0) = x_o, \tag{1.8}$$

the fractional-order law of mass action (second-order chemical reaction) [10]

$$D^{\alpha}x(t) \;=\; \rho\, (a_1 - x(t))\, (a_2 - x(t)), \quad \rho, \quad t > 0 \quad \text{and} \quad x(0) = x_o, \tag{1.9}$$

the fractional-order law of mass action (third-order chemical reaction) [10]

$$D^{\alpha}x(t) \;=\; \rho\, (a_1 - x(t))\, (a_2 - x(t))\, (a_3 - x(t)), \quad \rho, \quad t > 0 \quad \text{and} \quad x(0) = x_o, \tag{1.10}$$

and the fractional-order Stefan's law of radiation

$$D^{\alpha}x(t) \;=\; \rho\,(x^4(t)-a),\quad \rho,\quad t>0 \quad \text{and} \quad x(0)=x_o, \tag{1.11}$$

The existence of a unique positive solution $x \in C[0,T]$ of the problem (1.5)-(1.6) (under certain conditions) will be proved.
The stability of the solution of the problem (1.5)-(1.6) will be studied, also we prove that the equilibrium points of equation (1.5) are the same as the ones of the differential equation

$$\frac{d}{dt}x(t) = \sum_{k=0}^{n} a_k(t)\; f_k(x(t)).$$

As applications, the initial value problems (1.3), (1.4) and (1.8) -(1.10) will be studied.

2 Existence and uniqueness

Let $I=[0,T],\ T<\infty$ and $C(I)$, be the class of all continuous functions defined on I, with norm

$$||\ x\ || = \sup_t |\ e^{-Nt}x(t)\ |,\quad N>0 \tag{2.1}$$

which is equivalent to the sup-norm $||\ x\ || = \sup_t |\ x(t)\ |$. When $t>\sigma\geq 0$ we write $C(I_\sigma)$.
Let also $X=\{x\in L_1[0,T],\ e^{-Nt}x(t)\in L_1[0,T]\}$ with norm $||\ x\ ||_X = ||\ e^{-Nt}x(t)\ ||_{L_1}$ which is equivalent to the usual norm $||x||_{L_1} = \int_0^T |x(s)|ds$ of $L_1[0,T]$.

Consider now the initial value problem (1.5)-(1.6) with the following assumptions;
(1) $a_k(t)\in C^1[0,T],\ k=0,1,2,\cdots,$ the space of continuously differentiable functions on $I=[0,T],\ a_k>\sup|a_k(t)|$ and $a'_k>\sup|\frac{d}{dt}a_k(t)|$.
(2) $F: D \to R^+,\ \forall\, t\in I,\ D\subset R^+$ where $F(x(t))=\sum_{k=0}^{n} a_k(t)\ f_k(x(t)$.
(3) $\frac{\partial}{\partial\, x}f_k(x)$ exists and bounded on D.
Condition (3) implies that the functions f_k satisfy the Lipschitz condition

$$|f_k(x)-f_k(y)| < C_k\ |x-y|,\qquad C_k \geq\ |\frac{\partial}{\partial\, x}f_k(x)| \tag{2.2}$$

Now we have the following theorem

Theorem 2.1 *If the assumptions (1)-(3) are satisfied, then the initial value problem (1.5)-(1.6) has a unique positive solution* $x\in C(I)$, $x'\in C(I_\sigma)$ *and* $x'\in X$. *Moreover if* $a'_k(t)\neq 0$ *and* $a_k(0)=0$ *or* $\sum_{k=0}^{n} a_k(0)f(x_o)=0$, *then* $x'\in C(I)$.

Proof. From the properties of the fractional calculus and the problem (1.5)-(1.6) we have

$$I^{1-\alpha}\frac{d}{dt}x(t)=\sum_{k=0}^{n}\ a_k(t)\ f_k(x(t)).$$

Integrating $\alpha-$times we obtain

$$x(t)=x_o+I^{\alpha}\ \sum_{k=0}^{n}\ a_k(t)\ f_k(x(t)). \tag{2.3}$$

Now let the operator $A: C(I)\rightarrow C(I)$ be defined by

$$Ax(t)\ =\ x_o\ +\ I^{\alpha}\ \sum_{k=0}^{n}\ a_k(t)\ f_k(x(t)). \tag{2.4}$$

The operator A transforms every positive function $x\in C(I)$ into a function of the same type.
Now we can obtain

$$|e^{-Nt}(Ax-Ay)|\ \leq\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ e^{-N(t-s)}\ \sum_{k=0}^{n}|a_k(t)|\ |e^{-Ns}(f_k(x(s))-f_k(y(s))|\ ds$$

$$\leq\ K\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ e^{-N(t-s)}\ |e^{-Ns}(x(s)-y(s)|ds, \qquad K>\sum_{k=0}^{n}\ C_k\ |a_k(t)|$$

from which we obtain

$$|e^{-Nt}(Ax-Ay)|\ \leq\ ||x-y||\ K\ \int_0^t \frac{s^{\alpha-1}e^{-Ns}}{\Gamma(\alpha)}ds\ <\ \frac{K}{N^{\alpha}}\ ||x-y||.$$

Choose N such that $N^{\alpha}\ >\ K$ we deduce that

$$||Ax-Ay||\ <\ ||x-y||$$

and the operator A has a unique fixed point. Consequently the integral equation (2.3) has a unique positive solution $x\in C(I)$. Also we can deduce that ([6])

$$(I^{\alpha}\sum_{k=0}^{n}\ a_k(t)\ f_k(x(t)))|_{t=0}\ =\ 0.$$

Now from Eq. (2.3) we formally have

$$\frac{d}{dt}x(t)=\sum_{k=0}^{n}\{\ a_k(0)f_k(x_o)\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}+I^{\alpha}\ a_k(t)\ \frac{\partial}{\partial x}f_k(x(t))\ \frac{d}{dt}x(t)+\frac{d}{dt}a_k(t)f_k(x(t))\ \}$$

and

$$| e^{-Nt} \frac{d}{dt}x(t) | < \sum_{k=0}^{n}\{| a_k(0)f_k(x_o) | e^{-Nt} \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}+a'_k \int_0^t \frac{(t-s)^{\alpha-1} \ e^{-N(t-s)}}{\Gamma(\alpha)} |e^{-Ns}f_k(x(s))|ds\}$$

$$+ K \int_0^t \frac{(t-s)^{\alpha-1} \ e^{-N(t-s)}}{\Gamma(\alpha)} |e^{-Ns} \frac{d}{dt}x(s)| \ ds, \tag{2.5}$$

then

$$||\frac{d}{dt}x(t)|| < \sum_{k=0}^{n}\{ a_k(0)f_k(x_o) \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)} + a'_k || f_k(x(t)) || \} + \frac{K}{N^\alpha} ||\frac{d}{dt}x(t)|| \Rightarrow$$

$$||\frac{d}{dt}x(t)|| < \frac{1}{1-\frac{K}{N^\alpha}} \sum_{k=0}^{n}\{ a_k(0)f_k(x_o) \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)} + a'_k || f_k(x(t)) || \}$$

from which we deduce that $x' \in C(I_\sigma)$.
Now if $a_k(0) = 0$ or $\sum_{k=0}^{n} a_k(0)f(x_o) = 0$, then from (2.5) we obtain

$$||\frac{d}{dt}x(t)|| < \frac{1}{1-\frac{K}{N^\alpha}} \sum_{k=0}^{n} a'_k || f_k(x(t)) ||, \quad t \in I \quad \Rightarrow \quad x' \in C(I).$$

Also from (2.5) we can get

$$||\frac{d}{dt}x(t)||_X < \frac{1}{1-\frac{K}{N^\alpha}} \sum_{k=0}^{n}\{ a_k(0)f_k(x_o) + a'_k || f_k(x(t)) ||_X \}\frac{1}{N^\alpha} + \frac{K}{N^\alpha} ||\frac{d}{dt}x(t)||_X$$

which implies that $x' \in X$.
Now let $x(t)$ be the solution of the integral equation (2.3), then we have

$$x(t)|_{t=0} = x_o + (I^\alpha \sum_{k=0}^{n} a_k(t) \ f_k(x(t)))|_{t=0} = x_o.$$

and

$$D^\alpha x(t) = I^{1-\alpha}\frac{d}{dt}x(t) = I^{1-\alpha}\frac{d}{dt} \ I^\alpha \sum_{k=0}^{n} a_k(t) \ f_k(x(t)) =$$

$$\frac{d}{dt} \ I^{1-\alpha}I^\alpha \sum_{k=0}^{n} a_k(t) \ f_k(x(t)) = \sum_{k=0}^{n} a_k(t) \ f_k(x(t))$$

which proves the equivalence between the integral equation (2.3) and the initial value problem (1.5)-(1.6) and completes the proof of the theorem.
Consider now the initial value problem (1.7). Let $D = \{x \in R \ : 0 < x \leq b\}$ and $f_k(x(t)) = x^k(t)$, then we have

$$|\sum_{k=1}^{n} a_k(t) \ kx^{k-1}(t)| \leq \sum_{k=1}^{n} a_k \ kb^{k-1} = K, \quad \text{on D} \ , \ \forall \ t \in I.$$

Applying Theorem 2.1 we can prove the following corollary;

Corollary 2.2 *The initial value problem (1.7) (consequently the problem (1.8)) has a unique positive solution* $x \in C(I)$, $x' \in C(I_\sigma)$ *and* $x' \in X$. *If* $a'_k(t) \neq 0$ *and* $a_k(0) = 0$ *or* $\sum_{k=0}^{n} a_k(0)x_o^k = 0$, *then* $x' \in C(I)$.

Now let $a_k(t) = a_k$ (independent of t) in (1.7), then we have the following corollary of Theorem 2.1.

Corollary 2.3 *Each of the initial value problems (1.9)-(1.11) has a unique positive solution* $x \in C(I)$, $x' \in C(I_\sigma)$ *and* $x' \in X$.

3 Lyapunov uniform stability

Consider the initial value problem (1.5)-(1.6).

Definition 3.1 *The solution of the problem (1.5)-(1.6) is stable if,* $\forall\ \epsilon > 0$ *and* $t_o > 0$, *there exists* $\delta(\epsilon, t_o) > 0$ *such that for* $t \geq t_o$

$$|| \ x_o - x_o^* \ || < \delta(\epsilon, t_o) \quad \Rightarrow \quad || \ x(t) - x^*(t) \ || < \epsilon.$$

If δ *depends only on* ϵ, *then the solution is uniformly stable, where* $x^*(t)$ *is the solution of the initial value problem*

$$D^\alpha x(t) = \sum_{k=0}^{n} a_k(t) \ f_k(x(t)).,\quad t > 0, \quad and \ \ x(0) = x_o^*. \tag{3.1}$$

Now we have the following theorem;

Theorem 3.2 *The solution of the initial value problem (1.5)-(1.6) is uniformly stable.*

Proof. Let $x(t)$ and $x^*(t)$ are the solutions of the problems (1.5)-(1.6) and (3.1) respectively. Then we can get

$$|| \ x(t) - x^*(t) \ || \ \leq \ || \ x_o - x_o^* \ || + \frac{K}{N^\alpha} || \ x(t) - x^*(t) \ || \quad \Rightarrow$$

$$|| \ x(t) - x^*(t) \ || \ \leq \ \frac{1}{1 - \frac{K}{N^\alpha}} \ || \ x_o - x_o^* \ ||, \quad K < N^\alpha,$$

from which (by definition 3.1) we deduce that the solution of the problem (1.5)-(1.6) is uniformly stable and the theorem is proved.

Now let $f_k(x(t)) = x^k(t)$, then the following corollary can be proved;

Corollary 3.3 *The solution of the problem (1.7) consequently the solutions of the problems (1.3), (1.4) and (1.8)-(1.11) are uniformly stable.*

4 Equilibrium points and local stability

Consider the initial value problem

$$\frac{d}{dt}x(t) = F(x(t)) \;=\; \sum_{k=0}^{n} a_k(t)\; f_k(x(t)), \quad t > 0 \;\text{ and }\; x(0) = x_o. \tag{4.1}$$

To evaluate the equilibrium points of (4.1) let

$$\frac{d}{dt}x(t) = 0,$$

then the equilibrium points of the problem (4.1) are the solutions of the algebraic equation

$$F(x_{eq}) = 0.$$

To evaluate the asymptotic stability, let

$$x(t) = x_{eq} + \varepsilon(t), \quad \text{then } \frac{d}{dt}(x_{eq} + \varepsilon) = F(x_{eq} + \varepsilon) \Rightarrow \frac{d}{dt}\,\varepsilon(t) = F(x_{eq} + \varepsilon)$$

but

$$F(x_{eq} + \varepsilon) \;\simeq\; F(x_{eq}) + F'(x_{eq})\,\varepsilon + \;\cdots\; \Rightarrow \; F(x_{eq} + \varepsilon) \;\simeq\; F'(x_{eq})\,\varepsilon$$

where $F(x_{eq}) = 0$, then

$$\frac{d}{dt}\,\varepsilon(t) = F'(x_{eq})\,\varepsilon(t), \; t > 0, \;\text{ and }\; \varepsilon(0) = x_o - x_{eq}. \tag{4.2}$$

Now let the solution $\varepsilon(t)$ of (4.2) be exists. So if $\varepsilon(t)$ is increasing, then the equilibrium point x_{eq} is unstable and if $\varepsilon(t)$ is decreasing, then the equilibrium point x_{eq} is locally asymptotically stable.

Now we have the following theorem;

Theorem 4.1 *Let the solution $x \in C(I)$ of the initial value problem (1.5)-(1.6) be exists. Then the equilibrium points of the problem (1.5)-(1.6) are the same as the ones of the problem (4.1) i.e., are the solutions of the algebraic equation*

$$F(x(t)) \;=\; \sum_{k=0}^{n} a_k(t)\; f_k(x(t)) = 0$$

Proof. Consider the differential equation

$$D^\alpha x(t) \;=\; F(x(t)).$$

Then from the properties of the fractional integration we have

$$I^{1-\alpha}\frac{d}{dt}x(t) = F(x(t)) \;\Rightarrow\; I\frac{d}{dt}x(t) = I^\alpha F(x(t)) \;\Rightarrow\; \frac{d}{dt}x(t) = \frac{d}{dt}I^\alpha F(x(t)).$$

Also (note that $I^\alpha F(x(t))|_{t\ =0}\ = 0$) we have

$$\frac{d}{dt}x(t) = \frac{d}{dt}I^\alpha F(x(t)) \;\Rightarrow\; I^{1-\alpha}\frac{d}{dt}x(t) = I^{1-\alpha}\frac{d}{dt}I^\alpha F(x(t)) = \frac{d}{dt}I^{1-\alpha+\alpha}F(x(t)) = F(x(t)).$$

Then we deduce that the two differential equations

$$D^\alpha x(t) \;=\; F(x(t)) \quad \text{and} \quad \frac{d}{dt}x(t) \;=\; \frac{d}{dt}I^\alpha F(x(t))$$

are equivalent.
So the equilibrium points of the problem (1.5)-(1.6) are the solutions of the algebraic equation

$$\frac{d}{dt}x(t) = \frac{d}{dt}I^\alpha F(x(t)) = 0 \;\Rightarrow\; I^\alpha F(x(t)) = constant = C.$$

But

$$0 = I^\alpha F(x(t))|_{t\ =0} = C|_{t\ =0} \;\Rightarrow\; C = 0 \;\Rightarrow\; I^\alpha F(x(t)) = 0 \;\Rightarrow$$

$$IF(x(t)) = I^{1-\alpha}I^\alpha F(x(t)) = I^{1-\alpha}\ 0 = 0 \quad\Rightarrow\quad \int_0^t F(x(s))\ ds = 0 \quad\Rightarrow\quad F(x) = 0.$$

and the equilibrium points of the problem (1.5)-(1.6) are the solutions of the algebraic equations

$$F(x_{eq}) \;=\; 0$$

which completes the proof.

Consider now the initial value problem (1.5)-(1.6)

$$D^\alpha x(t) = F(x(t)) = \sum_{k=0}^{n} a_k(t)\ f_k(x(t)), \quad t > 0 \;\; \text{and} \;\; x(0) = x_o.$$

To evaluate the equilibrium points let

$$F(x(t)) = 0, \qquad \text{then} \quad F(x_{eq}) = 0.$$

To evaluate the asymptotic stability, let

$$F(t) = x_{eq} +\ \varepsilon(t), \quad \text{then} \;\; D^\alpha(x_{eq} +\ \varepsilon) = F(x_{eq} +\ \varepsilon) \;\;\Rightarrow\;\; D^\alpha\ \varepsilon(t) = F(x_{eq} +\ \varepsilon)$$

but

$$F(x_{eq}+\ \varepsilon)\ \simeq\ F(x_{eq})+F'(x_{eq})\ \varepsilon+\ \ \cdots\ \Rightarrow F(x_{eq}+\ \varepsilon)\ \simeq\ F'(x_{eq})\ \varepsilon$$

where $F(x_{eq})=0$, then

$$D^{\alpha}\ \varepsilon(t)=F'(x_{eq})\ \varepsilon(t),\ t>0,\ \text{ and }\ \varepsilon(0)=x_o-x_{eq}. \tag{4.3}$$

Now let the solution $\varepsilon(t)$ of (4.3) be exists. So if $\varepsilon(t)$ is increasing, then the equilibrium point x_{eq} is unstable and if $\varepsilon(t)$ is decreasing, then the equilibrium point x_{eq} is locally asymptotically stable.

4.1 Law of mass action (second-order chemical reaction)

For the law of mass action (second-order chemical reaction) we find that the equilibrium points are $a_1,\ ,a_2$. If $a_1\ <\ a_2\ \leq\ b$, we find that the local stability at $x_{eq}=a_1$ is asymptotical and at $x_{eq}=a_2$ is unstable. If $b\ \geq\ a_1\ >\ a_2$, we find that the local stability at $x_{eq}=a_1$ is unstable and at $x_{eq}=a_2$ is asymptotical.

4.2 Law of mass action (third-order chemical reaction)

For the law of mass action (third-order chemical reaction we find that the equilibrium points are a_1 , a_2, a_3. If $a_1\ <\ a_2\ <\ a_3\ \leq\ b$ or $b\ \geq\ a_1\ >\ a_2\ >\ a_3$, we find that the local stability at $x_{eq}=a_1$ is asymptotical, at $x_{eq}=a_2$ is unstable and at $x_{eq}=a_3$ is asymptotically stable.

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