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A nonlocal problem of an arbitrary (fractional) orders differential equation

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Abstract

In this paper we study the existence of solution for the differential equation of arbitrary (fractional) orders $\frac{dx}{dt} = f(t, D^{\alpha}x(t)),\ t \in (0,1)$ with the nonlocal condition $x(0) + \sum_{k=1}^{m} a_k\, x(t_k) = x_o$ where f is L^1-Caratheodory. The nonlocal integral condition $x(0) + \int_o^1 x(s)\, ds = x_o$ will be studied.

Keywords: Fractional calculus, nonlocal condition, integral condition, Caratheodory theorem.

1 Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1]), ([2]) and references therein.
Recently, El-Sayed, Abd El-Salam [4] studied the existence of a unique solution of the fractional order differential equation

$$D^{\alpha}x(t) = c(t)f(x(t)) + b(t), \quad t \in (0,1] \text{ and } \alpha \in (0,1]$$

with the nonlocal condition

$$x(0) + \sum_{k=1}^{m} a_k\, x(t_k) = x_o.$$

where $x_o \in \Re$, $0 < t_1 < t_2 < \ldots < t_m < 1$, $a_k \neq 0$, $k = 1, 2, \ldots, m$ and D^{α} is the fractional order operator.
In this work we study the existence of at least one solution for the nonlocal problem of the arbitrary (fractional) order differential equation

$$\frac{dx}{dt} = f(t, D^{\alpha}x(t)), \quad t \in (0,1] \text{ and } \alpha \in (0,1] \tag{1}$$

$$x(0) + \sum_{k=1}^{m} a_k\, x(t_k) = x_o, \quad t_k \in (0,1] \tag{2}$$

when f is L^1-Caratheodory.
As an application, we deduce the existence of solution for the nonlocal problem of the differential (1) with the nonlocal integral condition

$$x(0) + \int_0^1 x(s)\, ds = x_o. \tag{3}$$

2 preliminaries

Let $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [0,1]$, with the norm $||u||_{L^1} = \int_I |u(t)|\, dt$ and $C(I)$ denotes the class of continuous functions on the interval I , with the norm $||u|| = \sup_{t\in I} |u(t)|$ and $\Gamma(.)$ denotes the gamma function.

Definition 2.1 The fractional-order integral of the function $f \in L_1[a,b]$ of order $\beta \in R^+$ is defined by (see [6]- [9])

$$I_a^\beta f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s)\, ds$$

Definition 2.2 The Caputo fractional-order derivative of order $\alpha \in (0,1]$ of the absolutely continuous function $f(t)$ is defined by (see [7]-[9]).

$$D_a^\alpha f(t) = I_a^{1-\alpha} \frac{d}{dt} f(t).$$

Definition 2.3 The function $f : [0,1] \times R \to R$ is called L^1-Caratheodory if
(i) $t \to f(t,x)$ is measurable for each $x \in R$,
(ii) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$,
(iii) there exists $m \in L^1([0,1], D)$, $D \subset R$ such that $|f| \leq m$.
Now we state Caratheodory Theorem (see[3]).

Theorem 2.1 Let $f[0,1] \times R \to R$ be L^1-Caratheodory, then the initial-value problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad \text{for } a.e.\ t > 0, \quad \text{and} \quad x(0) = x_0. \tag{4}$$

has at least one solution $x \in AC[0,T]$.
Here we generalize Caratheodory theorem for the nonlocal problem (1)-(2).

3 Main results

Consider firstly the fractional-order integral equation

$$y(t) = I^{1-\alpha} f(t, y(t)), \tag{5}$$

Definition 3.1 The function y is called a solution of the fractional-order integral equation (5), if it is continuous on $[0,1]$ and satisfies (5).

Theorem 3.1 Let $f : [0,1] \times R \rightarrow R$ be L^1-Caratheodory. If $I_a^{1-\alpha}\ m(t) \leq M$, for $a \geq 0$, then there exists at least one solution $y \in C[0,1]$ of the fractional-order functional integral equation (5).
Proof. Since $I_a^{1-\alpha}\ m(t) \leq M$, then

$$\begin{aligned} \mid I_a^{1-\alpha}\ f(t, y(t)) \mid &\leq \int_a^t \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} \mid f(s, y(s)) \mid ds \\ &\leq \int_a^t \frac{(t - s)^{1-\alpha}}{\Gamma(1-\alpha)}\ m(s)\ ds \leq M, \quad a \geq 0. \end{aligned}$$

Define the sequence $\{y_n(t)\}$, $t \in [0,1]$

$$y_{n+1}(t) = \int_0^t \frac{(t - s)^{-\ \alpha}}{\Gamma(1 - \alpha)}\ f(s, y_n(s))\ ds,$$

which can be written in the operator form

$$y_{n+1}(t) = I^{1-\alpha\ -\ \beta}\ I^{\beta}\ f(t), \ \ y_n(t)).$$

Then

$$\begin{aligned} \mid y_{n+1}(t) \mid &\leq I^{1\ -\ \alpha\ -\ \beta} \mid I^{\beta}\ f(t,\ \ y_n(t))| \leq M \int_0^t \frac{(t - s)^{-\ \alpha\ -\ \beta}}{\Gamma(1-\alpha\ -\ \beta)}\ ds \\ &\leq M\ \frac{(t)^{1\ -\ \alpha\ -\ \beta}}{\Gamma(2\ -\ \alpha\ -\ \beta)} \leq \frac{M\ b^{1\ -\ \alpha\ -\ \beta}}{\Gamma(2\ -\ \alpha\ -\ \beta)}. \end{aligned}$$

For $t_1,\ t_2 \in [0,1]$ such that $t_1 < t_2$, then

$$\begin{aligned} y_{n+1}(t_2) - y_{n+1}(t_1) &= \int_0^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds - \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds \\ &= \int_0^{t_1} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds \\ &- \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds \\ &\leq \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s)\ ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds \\ &- \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s, y_n(s))\ ds. \end{aligned}$$

Therefore

$$\mid y_{n+1}(t_2) - y_{n+1}(t_1) \mid \leq \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)}\ m(s)\ ds \leq \int_{t_1}^{t_2} \frac{(t_2 - \theta)^{-\alpha}}{\Gamma(1-\alpha)}\ m(\theta)\ d\theta$$

$$\leq \quad M \int_{t_1}^{t_2} \frac{(t_2 \ - \ \theta)^{-\alpha-\beta}}{\Gamma(1-\alpha \ - \ \beta)} \, d\theta$$

$$\leq \quad M \, \frac{(t_2 \ - \ t_1)^{1-\alpha \ - \ \beta}}{\Gamma(2-\alpha \ - \ \beta)}.$$

Hence $\mid t_2 - t_1 \mid \ < \ \delta \quad \Rightarrow \quad \mid y_{n+1}(t_2) \ - \ y_{n+1}(t_1) \mid \ < \ \epsilon(\delta)$ and $\{y_n(t)\}$ is a sequence of equi-continuous and uniformly bounded functions. By Arzela-Ascoli Theorem, there exists a subsequence $\{y_{n_k}(t)\}$ of continuous functions which converges uniformly to a continuous function y as $k \ \to \ \infty$.
Now we show that this limit function is the required solution.
Since

$$\mid f(s, y_{n_k}(s)) \mid \ \leq \ m(s) \ \in \ L_1,$$

and $f(s, y_{n_k}(s))$ is continuous in the second argument,

$$\text{i.e. } f(s, y_{n_k}(s)) \ \to \ f(s, y(s)) \quad \text{as} \quad k \ \to \ \infty,$$

therefore the sequence $\{(t \ - \ s)^{- \ \alpha} \ f(s, y_{n_k}(s))\}$, $\alpha \ \in \ (0,1)$ satisfies Lebesgue dominated convergence theorem. Hence

$$\lim_{k \ \to \ \infty} \int_0^t \frac{(t \ - \ s)^{- \ \alpha}}{\Gamma(1 \ - \ \alpha)} \ f(s, y_{n_k}(s)) \ ds \ = \ \int_0^t \frac{(t \ - \ s)^{- \ \alpha}}{\Gamma(1 \ - \ \alpha)} \ f(s, y(s)) \ ds \ = \ y(t).$$

Which proves the existence of at least one solution $y \ \in \ C[0,1]$ of the fractional-order functional integral equation (5).

Theorem 3.2 Let the assumptions of Theorem 3.1 are satisfied. Then nonlocal problem (1)- (2) has at least one positive solution $x \in C[0,1]$.
Proof. Consider the nonlocal fractional differential equation

$$\frac{dx}{dt} \quad = \quad f(t, D^\alpha x(t)),$$

$$x(0) \quad + \quad \sum_{k \ = \ 1}^{m} a_k \ x(t_k) \ = \ x_\circ.$$

Let $y(t) \ = \ D^\alpha x(t)$, then

$$y(t) \ = \ I^{1-\alpha} \frac{dx(t)}{dt}, \tag{6}$$

$$y(t) \ = \ I^{1-\alpha} f(t, y(t)). \tag{7}$$

and y is the solution of the fractional-order integral equation (5).
Operating by I^α on both sides of equation(6), we get

$$I^\alpha y(t) \ = \ I \frac{dx(t)}{dt} \ = \ x(t) - x(0) \ \Rightarrow$$

$$x(t) \;=\; x(0) \;+\; I^{\alpha}\, y(t). \tag{8}$$

Substituting for the value of $x(0)$ from (2), we get

$$x(t) \;=\; x_0 \;-\; \sum_{k=1}^{m} a_k\, x(t_k) \;+\; I^{\alpha}\, y(t) \tag{9}$$

and

$$x(t_k) \;=\; x_0 \;-\; \sum_{k=1}^{m} a_k\, x(t_k) \;+\; I^{\alpha}\, y(t)|_{t=t_k}. \tag{10}$$

Now from (9) and (10) we can get

$$(1 \;+\; \sum_{k=1}^{m} a_k)\, x(t) \;=\; x_0 \;-\; \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k} \;+\; (1 \;+\; \sum_{k=1}^{m} a_k)\, I^{\alpha}\, y(t).$$

Letting $a \;=\; (1 \;+\; \sum_{k=1}^{m} a_k)^{-1}$, we deduce that the nonlocal problem (1)-(2) transformed to the integral equation

$$x(t) \;=\; a\; (\; x_0 \;-\; \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k}) \;+\; I^{\alpha}\, y(t) \tag{11}$$

which, by Theorem 3.1, has at least one solution $x \in C[0,1]$.
Now

$$\sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k} \;\leq\; \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)$$

and

$$a\, \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k} \;\leq\; a\, \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)$$

which gives

$$a\, \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k} \;\leq\; I^{\alpha}\, y(t)$$

and the solution

$$x(t) \;=\; a\; (x_0 \;-\; \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k}) \;+\; I^{\alpha}\, y(t)$$

is positive.
Substituting from (7) into (11), we obtain

$$x(t) \;=\; a\; (\; x_0 \;-\; \sum_{k=1}^{m} a_k\, I^{\alpha}\, y(t)|_{t=t_k}) \;+\; \int_0^t f(s, y(s))\, ds. \tag{12}$$

Differentiating both sides of (12), we get

$$\frac{dx}{dt} \;=\; f(t, y(t)) \;=\; f(t, D^{\alpha} x(t)).$$

Putting $t = 0$, in (11), we get

$$x(0) = a\ (\ x_0\ -\ \sum_{k=1}^{m}\ a_k\ (x(t_k) - x(0)))$$

and

$$x(0) = a\ (\ x_0\ -\ \sum_{k=1}^{m}\ a_k\ x(t_k) - x(0) \sum_{k=1}^{m}\ a_k)\ \ \Rightarrow$$

$$x(0)\ +\ x(0) \sum_{k=1}^{m}\ a_k\ =\ a\ (\ x_0\ -\ \sum_{k=1}^{m}\ a_k\ x(t_k)\ \ \Rightarrow$$

$$x(0) = a\ (1\ +\ \sum_{k=1}^{m}\ a_k)^{-1}(\ x_0\ -\ \sum_{k=1}^{m}\ a_k\ x(t_k)).$$

Since $a = (1\ +\ \sum_{k=1}^{m}\ a_k)$, it follows that

$$x(0) =\ x_0\ -\ \sum_{k=1}^{m}\ a_k\ x(t_k).$$

This completes the proof of the equivalence between the the integral equation (11) and the nonlocal problem (1)-(2).

4 Nonlocal integral condition

Let $x \in C[0, 1]$ be the solution of the nonlocal problem (1)-(2).
Let $a_k = \tau_k - \tau_{k-1}$, $t_k \in (\tau_{k-1}, \tau_k)$, $0\ =\ \tau_0 < \tau_1 < \tau_2, ... < \tau_n = 1$ then the nonlocal condition (2) will be

$$x(0)\ +\ \sum_{k=1}^{m}\ (\tau_k - \tau_{k-1})\ x(t_k)\ =\ x_o.$$

From the continuity of the solution x of the nonlocal problem (1)-(2) we can obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m}\ (\tau_k - \tau_{k-1})\ x(t_k)\ =\ \int_0^1\ x(s)\ ds.$$

and the nonlocal condition (2) transformed to the integral one

$$x(0)\ +\ \int_0^1\ x(s)\ ds\ =\ x_o.$$

Also from the continuity of the function $I^\alpha y(t)$, where y is the solution of the integral equation (5), we deduce that the solution (11) will be

$$x(t)\ =\ \frac{1}{2}\ (x_o\ -\ \int_0^1\ I^\alpha\ y(t)\ dt)\ +\ I^\alpha\ y(t).$$

Now, we have the following Theorem
Theorem 3.1 Let the assumptions of Theorem 3.2 are satisfied. Then there exist at least one solution $x \in C[0,1]$ of the nonlocal problem with integral condition,

$$\frac{dx(t)}{dt} = f(t, D^{\alpha}x(t)), \quad t \in (0,1],$$

$$x(0) + \int_0^1 x(s)\, ds = x_o.$$

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