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Caratheodory theorem for a nonlocal problem of the differential equation $x' = f(t, x')$

El-Sayed, A.M.A
Faculty of Science, Alexandria University, Alexandria, Egypt
amasayed@hotmail.com Elkadeky, Kh. W
Faculty of Science, Garyounis University, Benghazi, Libya
k-welkadeky@yahoo.com

Abstract

Here we are concerned with the existence of at least one solution of a nonlocal problem for the differential equation $x' = f(t, x')$, when f satisfy the assumptions of caratheodory theorem.

Key words: Caratheodory theorem, nonlocal conditions.

1 Introduction

The first-order three-point boundary value problems:

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ Mx(a) + Nx(b) + Rx(c) = \alpha. \end{cases}$$

was studied in ([8]) the existence and uniqueness of solutions for the three -point boundary value problems, where $f : [a, c] \times \Re^n \to \Re^n$ satisfies the Caratheodorys conditions, and M, N , and R are constant square matrices of order n and $\alpha \in \Re^n$. The existence of solutions is proven by the Leray-Schauder continuation theorem.
The nonlocal problem for first-order differential inclusion:

$$\begin{cases} \frac{dx(t)}{dt} \in F(t, x(t)), \ t \in (0, 1], \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0. \end{cases}$$

was studied in [1]and [6],where $F : J \times \Re \to 2^{\Re}$ is a set-valued, $J = [0, 1]$, $x_0 \in \Re$ is given, $0 < t_1 < t_2 < ... < t_m < 1,\ and\ a_k \neq 0,\ k = 1, 2, ..., m.$
The nonlocal problem for first-order differential equations:

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), a.e.\ t \in [0, 1], \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0. \end{cases}$$

was studied in [2] the existence of solutions for nonlinear first order differential equations with nonlocal conditions, where $f : [0,1] \times \Re \to \Re$ is a Caratheodory function, t_k are given points with $0 \leq t_1 \leq t_2 \leq ... \leq t_m < 1$ and a_k are real numbers with $1 + \sum_{k=1}^{m} a_k \neq 0$. Also the nonlocal problem for the fractional-order differential equation:

$$\begin{cases} D^{\alpha} x(t) \; = \; c(t) f(x(t)) \; + \; b(t), \; t \in (0,1], \\ \quad x(0) \; + \; \sum_{K=1}^{m} a_k x(t_k) \; = \; x_0. \end{cases}$$

has been studied in [5], where $x_0 \in \Re$ and $0 < t_1 < t_2 < ... < t_m < 1$, and $a_k \neq 0$ $k = 1, 2, ..., m$.

Consider the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad a.e, \; t > 0, \quad \text{and} \;\; x(0) = x_0. \tag{1}$$

Theorem(Caratheodory) see [3]

Let $f : [0,T] \times D \subset \Re \to \Re$ be measurable in t for any $x \in D$ and continuous in $x \in D$ for any $t \in [0,T]$. If there exists m in $L_1[0,T]$ such that $|f(t,x)| \leq m(t), \; (t,x) \in D$, then the problem (1) has at least one solution $x \in AC[0,T]$.

Here we generalize **Caratheodory Theorem** for the nonlocal problem

$$\frac{dx(t)}{dt} \quad = \quad f(t, \frac{dx(t)}{dt}), \; a.e, \; t \in \; (0,1], \tag{2}$$

$$x(0) \quad + \quad \sum_{k=1}^{m} a_k \, x(t_k) = x_0, \quad t_k \in (0,1]. \tag{3}$$

$x_o \in \Re, \; k = 1, 2, 3, ..., m$, $0 < t_1, t_2, ..., t_m < 1$.

The existence of at least one solution $x \in AC[0,1]$ will be studied when the function f satisfies the Caratheodory Theorem.

Also we deduce the existence of solution for the nonlocal problem for equation (2) with the nonlocal integral condition

In our proof we use the following two then

$$x(0) + \int_0^1 x(s) \; ds = x_o. \tag{4}$$

Theorem(Kolmogorov Compactness Criterion) see[4]

Let $\Omega \subseteq L^P(0,1), \; 1 \leq P < \infty$. If

(i) Ω is bounded $L^p(0,1)$,

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^P(0,1)$, where

$$x_h(t) = \frac{1}{h} \int_t^{t+h} x(s) \; ds.$$

Theorem(Schauder) see[7]
Let U be a convex subset of a Banach space X, and $T : U \to U$ is compact, continuous map. Then T has at least one fixed point in U.

2 Existence of solution

Consider firstly the functional equation

$$y(t) = f(t, y(t)),\ t \in (0, 1] \tag{5}$$

with the following assumptions :

(i) $f : [0, 1] \times D \subset \Re \to \Re$ is measurable in $t \in [0, 1]$, for any $y \in D \subset \Re$ and continuous in $y \in D$, for $t \in [0, 1]$.

(ii) There exists a function $m \in L_1[0, 1]$ such that :

$$|f(t, y)| \leq m(t)\ ;\ \forall (t, y) \in [0, 1] \times D\ .$$

(iii)

$$\int_0^1 m(t)\ dt \leq\ M\ ,\ M > 0.$$

Now we have the following theorem

Theorem 2.1 *Assume the assumption (i) - (iii) are satisfied. Then equation (5) has at least one solution $y \in L_1[0, 1]$.*

Proof. Define the operator H by:

$$Hy(t) = f(t, y(t)), \qquad t \in (0, 1]. \tag{6}$$

Let $y \in \Omega, \quad \Omega = \{y \in \Re\ :\ ||y|| < M\}$.
From assumption (i) and (iii), we obtain

$$\begin{aligned} ||Hy||_{L_1} &= \int_0^1 |(Hy)(t)|\ dt \\ &= \int_0^1 |f(t, y(t))|\ dt \\ &\leq \int_0^1 m(s)\ ds\ \leq\ M. \end{aligned}$$

Then $Hy \in \Omega$, which implies that the operator H maps Ω into itself, $\Omega \subseteq L_1[0, 1]$.
Assumption (ii) implies $f \in L_1[0, 1]$ and assumption (i) implies that H is continuous.

It remains to show that H is compact to apply **Schauder fixed point theorem**
Now, $\Omega \subseteq L_1[0,1]$, Ω is bounded in $L_1[0,1]$, therefore $H(\Omega)$ is bounded in $L_1[0,1]$, $i.e$ condition (i) of **Kolmogorav compactness criterion** is satisfied, it remains to show that $(Hy)_h \to (Hy)$, in $L_1[0,1]$
Let $y \in \Omega$, we have the following estimation :

$$\begin{aligned} ||(Hy)_h - (Hy)||_{L_1} &= \int_0^1 |(Hy)_h(t) - (Hy)(t)| \; dt \\ &= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Hy)(s) \; ds - (Hy)(t)| \; dt \\ &\leq \int_0^1 (\frac{1}{h}\int_t^{t+h} |(Hy)(s) - (Hy)(t)| \; ds) \; dt \\ &\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |f(s,y(s)) - f(t,y(t))| \; ds \; dt \end{aligned}$$

since $f \in L_1[0,1]$, it follows that:

$$\frac{1}{h}\int_t^{t+h} |f(s,y(s)) - f(t,y(t))| \; ds \;\to 0 \quad as \; h \to 0, \; for \; t \in [0,1]$$

Therefore:
$(Hy)_h \to (Hy)$, $uniformly \; as \; h \to 0$, Then by **Kolmogorav compactness criterion**, $H(\Omega)$ is relatively compact.
That is H has a fixed point in Ω, then there exist at least one solution $y \in L_1[0,1]$ such that $y(t) = f(t, y(t))$; $t \in [0,1]$.

Now, consider the nonlocal problem

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t, \frac{dx(t)}{dt}), a.e, \; t \;\in\; (0,1], \\ x(0) &+ \sum_{k=1}^{m} a_k \; x(t_k) = x_0, \quad t_k \in (0,1]. \end{aligned}$$

Theorem 2.2 *Let the assumption (i) - (iii) are satisfied. Then the nonlocal problem (2)-(3) has at least one solution* $x \in AC[0,1]$.

Proof. Let

$$\frac{dx(t)}{dt} = y(t), \qquad then \quad y(t) = f(t, y(t)) \tag{7}$$

and

$$x(t) = x(0) + \int_0^t y(s) \; ds. \tag{8}$$

Substituting for the value of $x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0$ into (8), we get

$$x(t) = x_0 - \sum_{k=1}^{m} a_k\ x(t_k) + \int_0^t y(s)\ ds. \tag{9}$$

If we $t = t_k$ in (9), we obtain

$$x(t_k) = x_0 - \sum_{k=1}^{m} a_k\ x(t_k) + \int_0^{t_k} y(s)\ ds \tag{10}$$

equation (9) and (10) implies that

$$x(t_k) = x(t) - \int_0^t y(s)\ ds + \int_0^{t_k} y(s)\ ds. \tag{11}$$

Substitute from (11) into (9), we get

$$x(t) = x_0 - \sum a_k[x(t) - \int_0^t y(s)\ ds + \int_0^{t_k} y(s)\ ds] + \int_0^t y(s)\ ds$$

$$= x_0 - \sum_{k=1}^{m} a_k\ x(t) + \sum_{k=1}^{m} a_k \int_0^t y(s)\ ds - \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds + \int_0^t y(t)\ ds. \tag{12}$$

which implies that

$$(1 + \sum_{k=1}^{m} a_k)x(t) = x_0 + (1 + \sum_{k=1}^{m} a_k) \int_0^t y(s)\ ds - \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds.$$

from which we obtain

$$x(t) = a(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds) + \int_0^t y(s)\ ds. \tag{13}$$

where $a = (1 + \sum_{k=1}^{m} a_k)^{-1}$
Now, form Theorem(2.1) and equation (13) we deduce that there exist at least one solution $x \in AC[0,1]$ of equation (13)
For complete the proof, we prove that equation (13) satisfies nonlocal problem (2)-(3).
Differentiating (13), we get

$$\frac{dx}{dt} = y(t) = f(t, \frac{dx}{dt})$$

Let $t = 0$ in (13) ,we get

$$x(0) = a(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds). \tag{14}$$

Let $t = t_k$ in (13) ,we get

$$x(t_k) = a(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds) + \int_0^{t_k} y(s)\ ds.$$

then

$$\sum_{k=1}^{m} a_k x(t_k) = \sum_{k=1}^{m} a_k a(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds) + \sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds\ . \qquad (15)$$

Addition (14) and (15) we obtain

$$x(0)+\sum_{k=1}^{m} a_k x(t_k) = a(x_0-\sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds)+\sum_{k=1}^{m} a_k a(x_0-\sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds)+\sum_{k=1}^{m} a_k \int_0^{t_k} y(s)\ ds.$$

$$= (1+\sum_{K=1}^{m} a_k)ax_0-a\sum_{k=1}^{m} a_k(1+\sum_{K=1}^{m} a_k)\int_0^{t_k} y(s)\ ds+\sum_{K=1}^{m} a_k \int_0^{t_k} y(s)\ ds.$$

since $a = (1 + \sum_{k=1}^{m} a_k)^{-1}$, therefor

$$x(0) + \sum_{k=1}^{m} a_k x(t_k) \ = \ x_0$$

this complete the proof of the equivalent between the nonlocal problem (2)-(3) and the integral equation (13) .
This implies that there exist at least one solution $x \in AC[0, 1]$ of the nonlocal problem (2)-(3).

3 Nonlocal integral condition

Let $x \in AC[0, 1]$ be the solution of the nonlocal problem (2)-(3).
Let $a_k = \eta_k - \eta_{k-1}, t_k \in (\eta_{k-1}, \eta_k),\ 0 = \eta_0 < \eta_1 < \eta_2, ... < \eta_n = 1$ then the nonlocal condition (3) will be

$$x(0) + \sum_{k=1}^{m} (\eta_k - \eta_{k-1})\ x(t_k) = x_o.$$

From the continuity of the solution x of the nonlocal problem (2)-(3) we can obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} (\eta_k - \eta_{k-1})\ x(t_k) = \int_0^1 x(s)\ ds.$$

$i.e$ the nonlocal condition (3) transformed to the integral one

$$x(0) + \int_0^1 x(s)\ ds = x_o$$

Now, we have the following corollary to Theorem(2.2).

Corollary 1 *Let the assumption of Theorem (2.2) are satisfied. If* $x(0) = x_0 = 0$, *then there exist at least one solution* $x \in AC[0,1]$ *of the nonlocal problem with integral condition,*

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t, \frac{dx(t)}{dt}), \ a.e, \ t \in \ (0,1], \\ \int_0^1 x(s)\ ds &= 0. \end{aligned}$$

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