Spectral Relationship of Cauchy Problem

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Abstract
In this paper we extend the applications of Schrödinger nuclear equation by using the perturbation theory and the basic theorems of integral equations to discuss the eigenvalues and eigenfunctions of linear integro- differential operator and consequently make a discussion on the stability of the solution.

1. Introduction
Cauchy problem and problems depending on Schrodinger equation have attracted the attention of many researchers. It is shown by means of the backlund transformation that the integrable initial boundary value problem on a semi-line for the nonlinear Schrodinger equation can be reduced to Cauchy problem for the same equation on the line [1]. Tabata and Eshima [2] gave an investigation of the equation of nonlinear partial differential equation blowing-up solutions to the Cauchy problem. A simple method for solving the Fredholm singular integro-differential equation with Cauchy kernel was proposed, based on a new reproducing kernel space [3]. Taylor- series expansion method and Galerkin method were used by Maleknejad and Arzhang [4] to obtain numerical solutions of the Fredholm singular integro-differential equation with Cauchy kernel.

In this paper, we discuss the asymptotic behavior of the eigenvalues and eigenfunctions of the following integro-differential equation

$$L\phi = -i\frac{d\phi}{dx} + p(x) + \int_a^b k(x,y)\phi(y)dy = \lambda\,\phi \qquad (1.1)$$
$$\phi(a) = \phi(b) = 1\,, i = \sqrt{-1}.$$

The operator L is known as a linear integro- differential operator and λ is a parameter. Equation (1.1), which takes the form $L\phi = \lambda\phi$, arises in many mathematical physics problems. It is often true that, the special solutions are called eigenfunctions or characteristic functions. These eigenfunctions must not be identically zero and satisfy one or more supplementary conditions related to the problem being solved. The eigenfunctions exist only for special values of the parameter λ; these values of λ are called eigenvalues or characteristic values.

We state some important lemmas and theorems, for the boundedness and orthogonality of the integro-differential operator of the Cauchy problem. The homogeneous integral equation with Schrödinger kernel is considered with its probability condition, some properties and relations for the Schrödinger kernel are stated and discussed. Furthermore, the iterated method is used to discuss the solution of the Schrödinger integral equation. We used the perturbation theory to obtain numerically the eigenvalues and eigenfunctions for the Schrödinger equation. Also, some different cases for the eigenvalues and the corresponding eigenfunctions are considered.

2. Boundedness and Orthogonality of the Integrodifferential Operator

Consider the boundary value of the integrodifferential equation (1.1) where $p(x)$ is a real continuous function in the interval (a,b), the kernel of the integral term $k(x,y)$ is continuous in the same interval. Moreover, $k(x,y)$ is symmetric, i.e.

$$k(x,y) = \overline{k(x,y)},$$

and λ is real.

Lemma-1

For the integrodifferential operator L of (1.1) and for every λ, the eigenfunctions of L under the conditions $|p(x)| = m_1$ and

$$\int_a^b \int_a^b |k(x,y)|^2 dxdy = m_2 < 1, \tag{2.1}$$

where m_1 and m_2 are constants, are bounded.

Proof

Write formula (1.1) in the form

$$\frac{d\phi}{dx} + i(p(x)-\lambda)\phi(x) = -i\int_a^b k(x,y)\phi(y)dy. \tag{2.2}$$

The solution of the ordinary differential equation (2.2), under the condition $\phi(a) = 1$, takes the form

$$\phi(x) = e^{i\int_a^x (\lambda - P(t))dt} - i\, e^{i\int_a^x (\lambda - P(t))dt} \int_a^x e^{i\int_a^\zeta (P(t)-\lambda)dt} \int_a^b k(\zeta,t)\phi(t)dtd\zeta. \tag{2.3}$$

Taking the norm of both sides of (2.3), and then using the two conditions of $p(x)$ and $k(x,y)$, we have

$$\|\phi\| \le \frac{1}{1-m_2}. \tag{2.4}$$

Formula (2.4) proves the boundedness of $\phi(x,\lambda)$ for all values of λ and $x \in [a,b]$.

Lemma-2: (without proof)

The eigenfunctions of the operator (1.1) corresponding to distinct eigenvalues are orthogonal.

Lemma-3: (without proof)

The eigenvalues of the integrodifferential operator (1.1) are real.

Theorem-1

Let $g(x)$ be an integrable function in the interval $[\alpha,\beta]$ and μ be a parameter, then

$$\int_\alpha^\beta e^{\pm i\mu x} g(x)dx \to 0 \qquad \text{as} \to \infty .$$

Theorem-2

For the boundary value problem of (1.1), the eigenvalues and eigenfunctions are asymptotically equivalent on the interval $[a, b]$ to the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{1}{i}\frac{d\phi}{dx} + p(x)\phi(x) = \lambda\phi(x)\,, \qquad \phi(a) = \phi(b) = 1$$

Proof

Write the solution of (1.1), after using the condition $\phi(a) = 1$ in the form

$$\phi(x) = e^{i\int_a^x (\lambda - p(t))dt} - i\, e^{i\int_a^x (\lambda - p(t))dt} \int_a^x e^{i\int_a^\zeta (p(t)-\lambda)dt} . \int_a^b k(\zeta, t)\phi(t)dtd\zeta \qquad (2.5)$$

Using the notations

$$\int_a^x p(t)dt = A(x)\ , \ \int_a^b k(\zeta, t)\phi(t)dt = F(\zeta) \qquad (2.6)$$

And then using the second condition $\phi(b) = 1,$ in (2.5) we get

$$1 = e^{i\lambda(b-a)-iA(b)} - ie^{i\lambda(b-a)-iA(b)} \int_a^b e^{-i\lambda(\zeta-a)} e^{iA(\zeta)} F(\zeta)d\zeta. \qquad (2.7)$$

The second term in the right hand side of (2.7) consists of the function

$$e^{i\lambda(b-a)-iA(b)},$$

which is bounded in the interval (a, b), also the function $e^{iA(\zeta)}\ F(\zeta), \zeta \in [a, b]$ is an integrable function in (a, b). Then by theorem (1) as $\lambda \to \infty$, the second term of (2.6) tends to zero. Thus for large value of λ, formula (2.7) becomes

$$1 = e^{i\int_a^b (\lambda - p(t))dt} + O(1). \qquad (2.8)$$

Therefore, for large value of λ the roots of (2.7) becomes

$$\lambda_m = \omega + \frac{2\pi i m}{b-a}\ , \omega = \frac{1}{b-a}\int_a^b p(t)dt; \quad m = 0, \pm 1, \pm 2, \dots\ , \qquad (2.9)$$

which can be adapted in the form

$$\lambda_m = \frac{2\pi i m}{b-a} + \ \omega + O(1), \qquad (2.10)$$

and the corresponding eigenfunctions are

$$\phi_m(x) = e^{i\int_a^x [(\omega - P(t)) + \frac{2\pi i m}{b-a}]dt} + O(1), \quad m = 0, \pm 1, \pm 2, \dots, \qquad (2.11)$$

which is different from the eigenfunctions of the differentiable operator

$$L_0 = \left(\frac{1}{i}\frac{d}{dx} + \ p(x)\right)$$

by small quantity tends to zero as $\lambda \to \infty$.

3. Homogenous Integral Equation

In this investigation Schrödinger obtained the integral equation

$$\lambda\phi(\boldsymbol{r}) = \int_{v'(r')} k(\boldsymbol{r},\boldsymbol{r}')\phi(\boldsymbol{r}')dv', \tag{3.1}$$

where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the position vectors of any two points of the body, $k(\boldsymbol{r},\boldsymbol{r}')$ is a function such that $k(\boldsymbol{r},\boldsymbol{r}')\,dv'$ is the probability that a neutron that is produced at the point $\boldsymbol{r}$ scores a splitting hit in the volume element dv' at $\boldsymbol{r}'$. The function $\phi(\boldsymbol{r})$ is the probability of a fission taking place at the point $\boldsymbol{r}$.

The whole reaction is started by the chance hit of a neutron on one of the nuclei of the atoms of the fissionable material. On fission this nucleus gives birth to a number, N say, of secondary neutrons. Each of these may score an efficient hit on some other nucleus and will do so unless it is removed earlier. This may occur through the escape of the neutrons from the surface of the reacting body. In dealing with a finite amount of fissionable material this leakage of neutrons from the surface may be appreciable. A neutron may also be caught in a non-splitting hit and get lodged in a nucleus. Such neutrons are evidently lost from the process since they do not directly produce any off-spring. Similarly any other elements presents as impurities may absorb neutrons and thereby reduce the number of secondary neutrons that, on the average, each primary neutron will produce.

We are only interested in the first eigenfunction $\phi(\boldsymbol{r})$ which is of constant sign and corresponding eigenvalue λ. From the physical interpretation of the function $\phi(\boldsymbol{r})$ we can normalize it by

$$\int p(\boldsymbol{r})dv = 1. \tag{3.2}$$

The importance of λ lies in the fact that for a symmetric $k(\boldsymbol{r},\boldsymbol{r}')$, it was shown by Schrödinger that λ sharply discriminates between the two cases:

(i) $< \frac{1}{N}$, no initial fission is sufficiently efficient to produce the chain reactions.

(ii) $> \frac{1}{N}$, any initial fission, wherever it occurs has a non vanishing chance of producing it.

Let us now find a reasonable expression for the kernel $k(\boldsymbol{r},\boldsymbol{r}')$ of the integral equation taking neutron absorption into consideration. If N neutrons are emitted at P, then since all directions are equally probable it follows that $N\frac{d\omega}{4\pi}$ is the number of neutrons emitted within a solid angle $d\omega$. On account of neutron absorption [2], the number crossing the element of surface $d\omega$ at Q may be reasonable taken as $N\frac{d\omega}{4\pi}e^{-ks}$, where k is an experimental constant of dimension L^{-1} called the absorption coefficient and $s = |\boldsymbol{r}-\boldsymbol{r}'|$. It depends on the kind of fissionable material with which we are dealing.

The number of neutrons crossing the element of surface at Q' is

$$N\frac{d\omega}{4\pi}e^{-k(s+ds)}$$

Hence, Schrödinger considered, the number absorbed in the showed volume as

$$N\frac{d\omega}{4\pi}e^{-ks} - N\frac{d\omega}{4\pi}e^{-k(s+ds)},$$

which may be adapted in the form

$$Nk\frac{d\omega}{4\pi}e^{-ks}ds$$

Of these only a fraction, μ say, scores splitting hits. Hence we can write

$$\mu Nk\frac{d\omega}{4\pi}e^{-ks}ds = Nk(\boldsymbol{r},\boldsymbol{r}')dv'$$

where dv' is the shaded volume
Hence, the required kernel of our integral equation is

$$k(\boldsymbol{r},\boldsymbol{r}') = \frac{A}{s^2}e^{-ks} \quad ,\left(A = \frac{\mu\kappa}{4\pi}\,,\ \frac{d\omega ds}{dv'} = \frac{1}{s^2}\right) \tag{3.3}$$

Hence, the eigenfunction $\phi(\boldsymbol{r})$ of the integral equation (3.12) with symmetric kernel (3.14) and corresponding eigenvalue , can be obtained. In this time, many different methods can be used to solve this problem analytically, see Knawel [5,6], Golberg [7] or numerically, see Atkinson[8] and Delves and Walsh [9].

4. The Weakly Nuclear Kernel

In the branch space $C(\partial D)$ of complex-valued continuous functions defined on the surface D, where D denote a bounded open region in R^3 and ∂D is the boundary of D equipped with the maximum norm $\|\phi\|_\infty = \max_{x\in D}|\phi(x)|$ we consider the integral operator
$A: C(\partial D) \to C(\partial D)$ defined by

$$(A\phi)(x) = \int_{\partial D} k(x,y)\phi(y)ds(y), \qquad x \in \partial D \tag{4.1}$$

A kernel $k(x,y)$ is said to be weakly singular if for all $x, y \in \partial D$ there exist positive constants M and $\alpha \in (0,2]$ such that

$$|k(x,y)| \leq M|x-y|^{\alpha-2}. \tag{4.2}$$

Theorem-3
The integral operator (4.1) with weakly singular kernel (4.2) is a compact operator.
Proof:
From (4.1) and using polar coordinate (ρ,θ) we can write

$$|A\phi(x)| \leq 2M\|\phi\|_\infty \int_0^{2\pi}\int_0^R \rho^{\alpha-1}d\rho d\theta \tag{4.3}$$

Hence

$$|A\phi(x)| \leq 2M\|\phi\|_\infty \frac{R^\alpha}{\alpha} \quad , \ \alpha \in (0,2] \tag{4.4}$$

Finally the operator A is bounded, therefore it is compact operator.
Let m be a complex number such that $\Im m\, m \geq 0$ then the kernel (nuclear kernel).

$$k(x,y) = \frac{e^{im|x-y|}}{4\pi|x-y|} \quad x, y \in R^3\,;\ x \neq y \tag{4.5}$$

is a solution to the Helmoholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + m^2\right)R(x,y) = 0, \tag{4.6}$$

with respect to$x$ for any fixed y. Because of its pole like singularity at $x = y$, the kernel function R is called a fundamental solution to the Helmoholtz equation.
Given a function $\phi \in C(\partial D)$, the function

$$u(x) = \int_{\partial D} R(x,y)\phi(y)ds(y) \qquad x \in R^3\partial D \tag{4.7}$$

is called the acoustic single-layer potential with densityϕ . The reader can easily see that $u(x)$ is a solution of the Helmoholtz equation.

Theorem-4

Let G be a closed domain containing ∂D in the interior. Assume the function $k(x,y)$ is defined and continuous for all $x \in G, y \in \partial D, x \neq y$ and it given by (4.5). Assume further that there exists $n \in N$ such that

$$|k(x_1,y) - k(x_2,y)| \le M \sum_{J=1}^{n} |x_1 - y|^{\alpha-2-J} |x_1 - x_2|^J \tag{4.8}$$

For all $x_1, x_2 \in G, y \in \partial D$ with $2|x_1 - x_2| \le |x_1 - y|$. Then the generalized potential u defined by

$$u(x) = \int_{\partial D} k(x,y)\phi(y)ds(y), \quad x \in G \tag{4.9}$$

with density $\phi \in C(\partial D)$ belongs to the Holder space $C^{0,\beta}(G)$ for all $\beta \in (0,\alpha]$ if $0 < \alpha < 1$, for all $\beta \in (0,2]$ if $\alpha = 1$, and for all $\beta \in (0,1]$ if $0 < \alpha < 2$ and

$$\|u\|_{\beta,G} \le C_\beta \|u\|_{\infty\partial D,} \tag{4.10}$$

for some constant C_β depending on.β.

Proof:

By the arguments used in Theorem 4 the function u is well defined as an improper integral for $x \in \partial D$. We can introduce parallel surfaces ∂D_n to ∂D by the representation

$$x = z + hv(z), z \in \partial D \tag{4.11}$$

where the parameter h denotes the distance of ∂D_n from the generating surface ∂D. Since ∂D is assumed to be of class C^2, we observe that ∂D_n is of class C' choose a positive number such that the parallel surfaces (4.11) are well defined for all $|h| \le h_0$ and define the set D_{h_0} by

$$D_{h_0} = \{x = z + hv(z), z \in \partial D, |\mathrm{h}| \le \mathrm{h}_0\}. \tag{4.12}$$

Then, analogous to (4.10), we can easily show that

$$|u(x)| \le C\|\phi\|_\infty \qquad \text{for all } x \in \partial D_n \tag{4.13}$$

To establish the uniform Holder continuity, let $x_1, x_2 \in \partial D_{h_0}$ with $0 < |x_1 - x_2| < \frac{R}{4}$.

Both x_1, x_2 may lie on ∂D .

Now choose uniquely points $z_1, z_2 \in \partial D$ such that $x_j = z_j + h_j v(z_j)$, $j = 1,2$.

Hence we have

$$\frac{1}{2}|x_1 - x_2| \le |z_1 - z_2| \le 2|x_1 - x_2| \tag{4.14}$$

i.e. here, we set

$$r = 4|x_1 - x_2| \tag{4.15}$$

Using (4.5) we follow

$$\int_{S_{z_1,r}} [k(x_1,y) - k(x_2,y)]\phi(y)ds(y)$$
$$\le M\|\phi\|_\infty \left\{ \int_{S_{z_1,r}} |x_1 - y|^{\alpha-2} ds(y) + \int_{S_{z_2,r}} |x_2 - y|^{\alpha-2} ds(y) \right\} \tag{4.16}$$

Using the fact that $S_{z_1,r} \sqsubset S_{z_2,r}$, see (4.14), and using polar coordinates, we have

$$\int_{S_{z_1,r}} [k(x_1,y) - k(x_2,y)]\phi(y)ds(y) \le C_1(M,\alpha)\|\phi\|_\infty |x_1 - x_2|^\alpha \tag{4.17}$$

Using condition (4.7) we have

$$\int_{S_{z_1,R},S_{z_1,r}} [k(x_1,y) - k(x_2,y)]\phi(y)ds(y) \le M\|\phi\|_\infty \sum_{j=1}^{\infty} |x_1 - x_2|^j \int_{S_{z_1,R},S_{z_1,r}} |x_1 - y|^{\alpha-2-j} ds(y) \le 4\pi M\|\phi\|_\infty \sum_{j=1}^{\infty} |x_1 - x_2|^J \int_{\frac{r}{4}}^{R} \rho^{\alpha-1-j} d\rho \quad (4.18)$$

where we have used the fact that the projection of $S_{z_1,R,}S_{z_1,r}$ into the tangent plane at z_1 is contained in the annulus with radii $\frac{r}{4}$ and R Now, we note that

$$\int_{\frac{r}{4}}^{R} \rho^{\alpha-1-j} d\rho \le \begin{cases} \frac{1}{j-\alpha} |x_1 - x_2|^{\alpha-j} & if\ j > \alpha \\ \ln \frac{R}{|x_1 - x_2|} & if\ j = \alpha \\ \frac{1}{\alpha-j} R^{\alpha-j} & if\ j < \alpha \end{cases} \quad (4.19)$$

and if $\beta \in (0,1)$, $|x_1 - x_2| < 1$ we have

$$|x_1 - x_2| \ln \frac{1}{|x_1 - x_2|} \le \frac{1}{1-\beta} |x_1 - x_2|^\beta \quad (4.20)$$

Hence,

$$\left| \int_{S_{z_1,R},S_{z_1,r}} [k(x_1,y) - k(x_2,y)]\phi(y)ds(y) \right| \le \begin{cases} C_2\|\phi\|_\infty |x_1 - x_2|^\alpha & \alpha < 1 \\ C_2\|\phi\|_\infty |x_1 - x_2|^\beta & \alpha = 1 \\ C_2\|\phi\|_\infty |x_1 - x_2| & \alpha > 1 \end{cases} \quad (4.21)$$

For some constant C_2 depending on m, M, R , α and β.
Finally, again using (4.7) we have

$$\int_{\partial DS_{z_1,R}} [k(x_1,y) - k(x_2,y)]\phi(y)ds(y) \le M\|\phi\|_\infty \sum_{j=1}^{\infty} |x_1 - x_2|^j \int_{\partial DS_{z_1,R}} |x_1 - y|^{\alpha-2-j} ds(y) \le C_3 M\|\phi\|_\infty |x_1 - x_2| \quad (4.22)$$

For some constants depending on m, M, R , α and $|\partial D|$.
Combining (4.32), (4.34), (4.36) and (4.37) we obtain

$$|u(x_1) - u(x_2)| \le (C_1 + C_2 + C_3)|x_1 - x_2|^\beta \|\phi\|_\infty \quad (4.23)$$

For all $x_1, x_2 \in D_{h_0}$ with $|x_1 - x_2| \le \frac{R}{4}$.Also, if $|x_1 - x_2| \ge \frac{R}{4}$, one can obtain

$$|u(x_1) - u(x_2)| \le 2C \left(\frac{4}{R}\right)^\beta |x_1 - x_2|^\beta \|\phi\|_\infty \quad (4.24)$$

For the kernel of (4.20) of the surface potential, for points on the boundary, we have

$$|k(x,y)| \le \frac{1}{4\pi} |x - y|^{-1} \quad (4.25)$$

Therefore, the kernel of Helmoholtz equation is weakly singular with $\alpha = 1$and hence, the single-layer potential is well defined for all points $x \in \partial D$. Using the inequality

$$||x_1 - y|^{-1} - |x_2 - y|^{-1}| \le |x_1 - x_2| |x_1 - y|^{-1} |x_2 - y|^{-1} \le 2 |x_1 - x_2| |x_1 - y|^{-2} \quad (4.26)$$

For $2|x_1 - x_2| \le |x_2 - y|$
and

$$\left| e^{im|x_1-y|} - e^{im|x_2-y|} \right| \le m|x_1 - x_2| \tag{4.27}$$

We observe that $k(x,y)$ satisfies (4.25) with $n = 1$.This results leads us to state the following.

Theorem-5

The single-layer potential u with continuous density ϕ is uniformly Hölder continuous throughout R^3 and $\|u\|_{\alpha,R^3} \le C(\alpha,\partial D)\|\phi\|_{\infty,\partial D}$ $(0 < \alpha < 1)$

5. Perturbation Theory in the Nuclear Integral Equation

In this section we try to discuss the solution of the nuclear integral equation, using the perturbation theory. For this, we have

$$\check{\lambda}\phi(r) = \frac{\mu k}{2}\int_{r'}^{a}\int_{\theta=0}^{\pi}\frac{e^{-k|r-r'|}}{(r'-r)^2}\phi(r')r'^2\sin\theta d\theta dr' \tag{5.1}$$

Changing the variable θ into s, we obtain

$$\check{\lambda}\phi(r) = \frac{\pi\mu k}{2}\int_0^a r'\phi(r')\int_{|r-r'|}^{r+r'}\frac{e^{-ks}}{s}ds dr' \tag{5.2}$$

Introducing, in (5.2), the function $\psi(r) = r\phi(r)$, then using the following notations $r = ax$, $r' = ax'$, $s = at$, we get

$$\lambda\psi(x) = \int_0^1 g(x,x')\,\psi(x')dx' \quad (\check{\lambda} = \lambda\mu\varepsilon), \tag{5.3}$$

where

$$g(x,x') = \frac{1}{2}\int_{|x-x'|}^{x+x'}\frac{e^{-\varepsilon t}}{t}dt \tag{5.4}$$

According to the perturbation theory, after expanding (5.4), we can write

$$g^{(0)}(x,x') = \frac{1}{2}\ln\frac{x+x'}{x-x'} = \begin{cases} \sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{x'}{x}\right)^{2k-1} & x' \le x \\ \sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{x'}{x}\right)^{2k-1} & x \le x' \end{cases} \tag{5.5}$$

This is the zero perturbed kernel, (coefficient of $\varepsilon^{(0)}$)

Moreover, for the first perturbation (coefficient of $\varepsilon^{(1)}$) we have

$$g^{(1)}(x,x') = -sm(x,x'), \tag{5.6}$$

where $sm(x,x')$ means the smaller of (x,x'). For the second to fourth perturbation, we write

$$g^{(2)}(x,x') = \frac{1}{2}xx' \tag{5.7}$$

$$g^{(3)}(x,x') = \begin{cases} -\frac{1}{9}\left(x'^2 + 3x^2x'\right) & x' \le x \\ -\frac{1}{9}(x^3 + 3x^2x') & x \le x' \end{cases} \tag{5.8}$$

and

$$g^{(4)}(x,x') = \frac{1}{12}xx'(x^2 + x'^2) \tag{5.9}$$

Let us try to solve (5.3) by setting

$$\psi(x) = \sum_{n=1}^{\infty}C_n x^{2n-1} \qquad , \psi(x) = xP(x) \tag{5.10}$$

where, the constant C_n can be expanded as

$$C_n = \sum_{j=0}^{\infty}C_n^{(j)}\varepsilon^j \tag{5.11}$$

Hence, we have

$$\lambda \sum_{n=1}^{\infty} C_n x^{2n-1} = \int_0^1 \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} g^{(j)}(x,x')\, \varepsilon^j C_n\, x'^{2n-1} dx' \qquad (5.12)$$

Substituting the values of the kernels into (5.12) and integrating the results to ε^2 , we have

$$\lambda \sum_{n=1}^{\infty} C_n x^{2n-1} = \sum_{j=0}^{\infty} \sum_{\nu=1}^{\infty} \frac{C_\nu^{2n-1}}{x}(2n-1)(2\nu-2n-1) + 2\sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty} C_\nu x^{2\nu} . \frac{1}{(2n-1)^2-4\nu^2} - \varepsilon \sum_{\nu=1}^{\infty} \frac{C_\nu}{2\nu}\left(x - \frac{x^{2\nu+1}}{2\nu+1}\right) + \frac{1}{2}\varepsilon^2 x \sum_{\nu=1}^{\infty} \frac{C_\nu}{2\nu+1} + \cdots \qquad (5.13)$$

It is easily to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-4\nu^2} = \frac{\pi^2}{8}\delta_{\nu 0} \qquad ,\nu = 0,1,2, \ldots \qquad (5.14)$$

where

$$\delta_{\nu 0} = \begin{cases} 1 & \nu = 0 \\ 0 & \nu \neq 0 \end{cases}$$

From (5.13) we deduce that the second term of the right hand side is vanished, where formula (5.14) represents the coefficients of even functions. This explains why we assumed that formula (5.10) is expanded in terms of odd functions. Equating the coefficient of x and x^{2n+1}, $k = 1,2, \ldots$; on both sides of (5.13), we get

$$\lambda C_1 = \sum_{n=1}^{\infty} \frac{C_n}{2n-1} - \frac{\varepsilon}{2}\sum_{\nu=1}^{\infty} \frac{C_\nu}{\nu} + \frac{\varepsilon^2}{2}\sum_{\nu=1}^{\infty} \frac{C_\nu}{2\nu+1} + \cdots \qquad (5.15)$$

$$(2n+1)\lambda C_{n+1} = \sum_{\nu=1}^{\infty} \frac{C_\nu}{2\nu-2n-1} + \frac{\varepsilon}{2}\frac{C_n}{n} \ , \ n = 1,2, \ldots \qquad (5.16)$$

Putting $\varepsilon = 0$ in (5.15) and (5.16), we get the recurrence relations

$$\lambda^0 C_n^0 = \sum_{\nu=1}^{\infty} \frac{C_\nu^0}{(2\nu-2n+1)(2n-1)} \qquad (5.17)$$

Formula (5.17) represents the recurrence relations for the non- perturbed integral equation

$$\lambda^0 \psi^0 = \int_0^1 g^{(0)}(x,x')\psi^{(0)}(x')d\,x' \qquad (5.18)$$

The solution of formula (5.17) is obtained after knowing the eigenvalues of the infinite matrix M, with elements

$$M_{ij} = \frac{1}{(2i-1)(2j-2i+1)} \qquad (5.19)$$

If we approximate the matrix given by (5.18) by letting i and j run from 1 to m only, we get

for $n = 3, \lambda^0 = 0.7924$, $P^0(x) = 1 - 0.6965x^2 + 0.1226x^4$
for $n = 4, \lambda^0 = 0.7862$, $P^0(x) = 1 - 0.7081x^2 + 0.1256x^4 - 0.0199x^6$
for $n = 5, \lambda^0 = 0.7852$, $P^0(x) = 1 - 0.7099x^2 + 0.1261x^4 - 0.0202x^6 - 0.0037x^8$ (5.20)

In dealing with the matrix (5.19), where i, j run from 1 to 20, if we repeat the iteration process in question seven times, the scalar quantity extracted and the corresponding are column matrix fairly constant and we get for them (for $n = 20, \lambda^0 = 0.7853$)

$$P^0(x) = 1 - 0.7099x^2 + 0.1261x^4 - 0.0202x^6 - 0.0037x^8 - 0.0035x^{10} - 0.0025x^{12} - 0.0019x^{14} - 0.0015x^{16} - 0.0012x^{18} - 0.0009x^{20} - 0.0008x^{22} - 0.0007x^{24} - 0.0006x^{26} - 0.0005x^{28} - 0.0005x^{30} - 0.0004x^{32} - 0.0004x^{34} - 0.0003x^{36} - 0.0003x^{38} \qquad (5.21)$$

To improve these results we apply the iteration method. This method starts from an approximate eigenfunction, for which we shall take from the last line of (5.20). Applying the static condition, with the aid of (5.10) and (5.11), we get

$$3V\sum_{n=1}^{\infty}\frac{C_n^{(0)}}{2n+1}=1,$$
$$\sum_{n=1}^{\infty}\frac{C_n^{(j)}}{2n+1}=0,\quad j=1,2,\dots,m \tag{5.22}$$

where m is the m^{th} perturbation, and V is the volume of the sphere. Hence, the normalized solution is

$$P^0(x)=\frac{1}{3V}\sum_{n=1}^{\infty}C_n x^{2n-2}. \tag{5.23}$$

The previous formula gives the distribution through the different points of the sphere of the probability of occurrence of a fission in the case of very small absorption.
In order to include absorption we now put in (5.15) and (5.16)

$$C_n=\sum_{j=0}^{\infty}\varepsilon^j C_n^{(j)}, \tag{5.24}$$

$$\lambda=\sum_{j=0}^{\infty}\varepsilon^j\lambda^{(j)} \tag{5.25}$$

For formula (5.15) we proceed as follows

$$\lambda^0 C_1^0=\sum_{\nu=1}^{\infty}\frac{C_\nu^{(0)}}{2\nu-1}$$
$$\lambda^{(0)}C_1^{(1)}+\lambda^{(1)}C_1^{(0)}=-\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{\nu}C_\nu^{(0)}+\sum_{\nu=1}^{\infty}\frac{C_\nu^{(1)}}{2\nu-1},$$
$$\lambda^{(0)}C_1^{(2)}+\lambda^{(1)}C_1^{(1)}+\lambda^{(2)}C_1^{(0)}=-\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\left(C_\nu^{(0)}+C_\nu^{(1)}\right)+\sum_{\nu=1}^{\infty}\frac{C_\nu^{(2)}}{2\nu-1} \tag{5.26}$$

Also,

$$(2k+1)\lambda^{(0)}C_{k+1}^{(0)}=\sum_{\nu=1}^{\infty}\frac{C_\nu^{(0)}}{2\nu-2k-1}$$
$$(2k+1)\left[\lambda^{(0)}C_{k+1}^{(1)}+\lambda^{(1)}C_{k+1}^{(0)}\right]=-\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{2\nu-2k+1}C_\nu^{(1)}+\frac{C_k^{(0)}}{2k}$$

$$(2k+1)\left[\lambda^{(0)}C_{k+1}^{(2)}+\lambda^{(1)}C_{k+1}^{(1)}+\lambda^{(2)}C_{k+1}^{(0)}\right]=\sum_{\nu=1}^{\infty}\frac{1}{2\nu-2k-1}C_\nu^{(2)}+\frac{C_k^{(1)}}{2k} \tag{5.27}$$

In the final form, we have

$$\sum_{m=0}^{N}\lambda^{(m)}C_k^{(N-m)}=\sum_{\nu=1}^{\infty}\frac{1}{2\nu-2k+1}C_\nu^{(N)}+\frac{C_k^{(N-1)}}{2k(2k-1)} \tag{5.28}$$

Hence, for the first and second perturbation of $\psi(x)$, we have

$$\left(M-\lambda^{(0)}I\right)\psi^{(j)}=a^{(j)},\qquad j=1,2 \tag{5.29}$$

Where M is the matrix (5.19), I is the unit matrix, $\psi^{(1)}$ and $\psi^{(2)}$ is the first and second order perturbation in $\psi^{(0)}$, and $a^{(1)}$, $a^{(2)}$ are column vectors whose components are

$$a_1^{(1)}=\lambda^{(1)}C_1^{(0)}+\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{\nu}C_\nu^{(0)}$$
$$a_{k+1}^{(1)}=\lambda^{(1)}C_k^{(0)}-\frac{C_k^{(0)}}{2k(2k-1)} \tag{5.30}$$
$$a_1^{(2)}=\lambda^{(1)}C_1^{(1)}+\lambda^{(2)}C_1^{(0)}-\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{2\nu+1}C_\nu^{(0)}+\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{\nu}C_\nu^{(1)}$$

$$a_{k+1}^{(2)} = \lambda^{(1)}C_{k+1}^{(1)} + \lambda^{(2)}C_{k+1}^{(0)} - \frac{c_k^{(1)}}{2k(2k+1)}, \quad k = 1,2,\ldots \tag{5.31}$$

Formulas (5.27) have a solution if and only if

$$\beta^{+}a^{(1)} = 0 \tag{5.32}$$

where the column vector β is determined by

$$(M^{+} - \lambda^{0}I)\beta = 0 \tag{5.33}$$

Therefore, in case of $n = 20$ the value is $\lambda^{0} = 0.78315763161771$.

The values of the constants $C_1^{(0)},\ C_2^{(0)},\ C_3^{(0)},\ldots,C_{20}^{(0)}$, are created from the relation $(M - I\lambda^{0})C_{ij}^{(0)} = \bar{0}$. The first five valuable piece of information was found when $n = 5$ and also the production of the constants $\beta_1^{(0)},\ \beta_2^{(0)},\ \beta_3^{(0)},\ldots,\beta_{20}^{(0)}$, from the relation $(M^{+} - I\lambda^{0})\beta_{ij}^{(0)} = \bar{0}$ where M^{+} is the transpose of the matrix M. Also, for the creation of the values of $a_1^{(1)},\ a_2^{(1)},\ a_3^{(1)},\ldots,a_{20}^{(1)}$, from equations (5.30) and (5.29) to find the values of the first perturbation $\psi^{(1)}(x)$ that follows, see Figures 1, 2 , 3 , 4, in accordance with

$$\psi^{(1)}(x) = (M - I\lambda^{1})^{-1}a^{(1)} \tag{5.34}$$

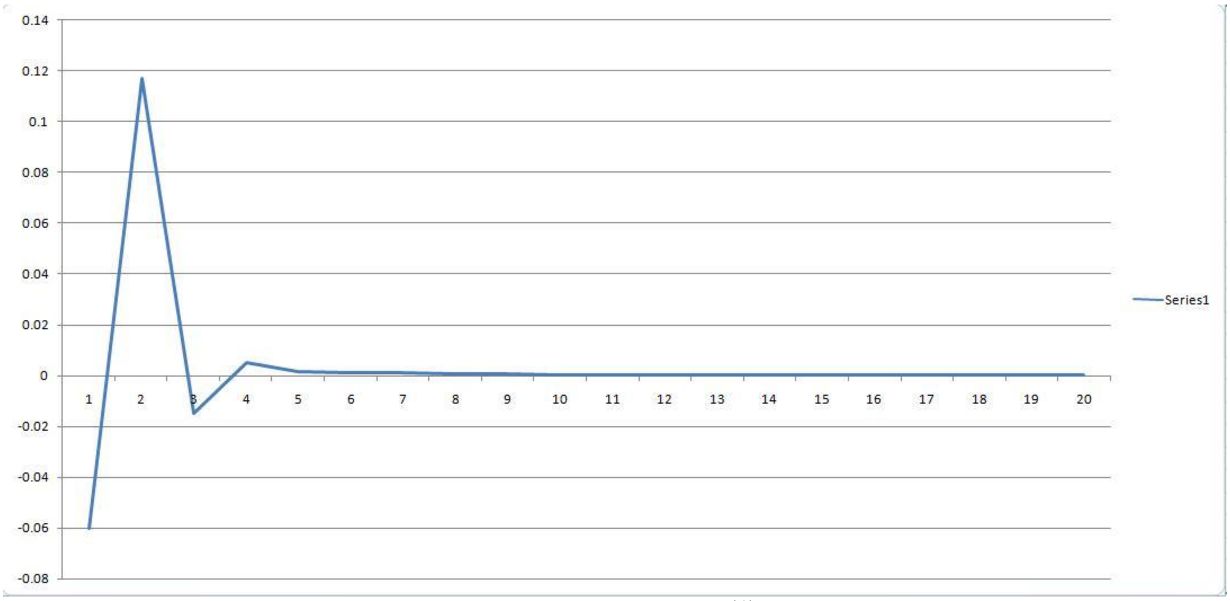

Figure 1: the relation between the values of $\boldsymbol{a}^{(1)}$

In the same way we created the values of $C_1^{(1)},\ C_2^{(1)},\ C_3^{(1)},\ldots,C_{20}^{(1)}$ and $\beta_1^{(0)},\ \beta_2^{(0)},\ \beta_3^{(0)},\ldots,\beta_{20}^{(0)}$ when $\lambda^{1} = -0.39979783324364$.
and also created the values $a_1^{(2)},\ a_2^{(2)},\ a_3^{(2)},\ldots,a_{20}^{(2)}$, see Figures 2, 3, and 4.

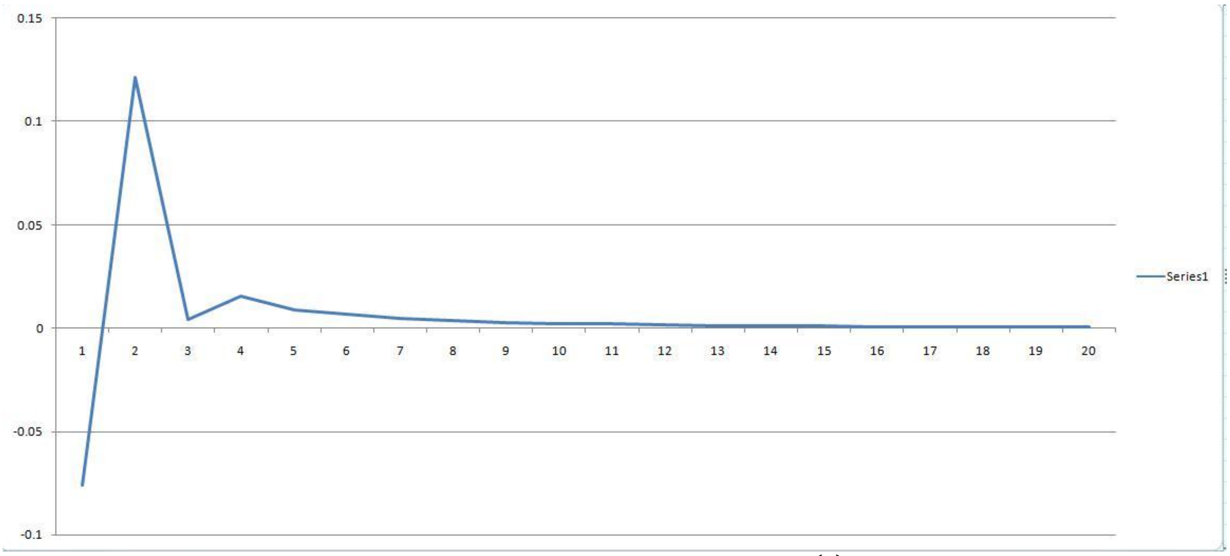

Figure 2: The relation between the values of $\psi^{(1)}$

second perturbation $\psi^{(2)}(x)$ when $\lambda^2 = -0.29216297148720$.

Thus

$$\psi^{(2)}(x) = (M - I\lambda^1)^{-1} a^{(2)} \tag{5.35}$$

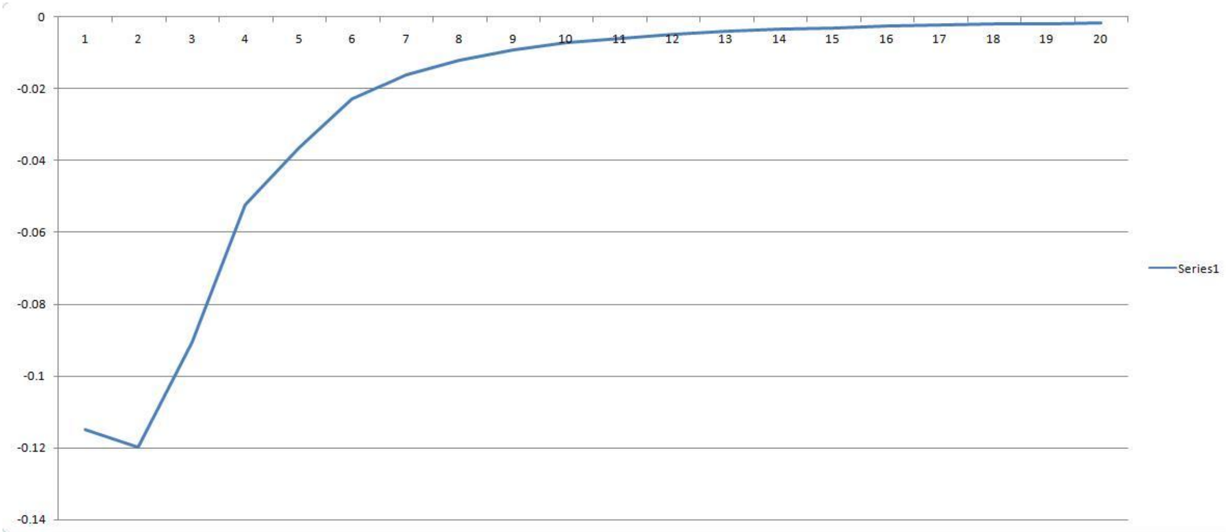

Figure 3: The relation between the values of $a^{(2)}$

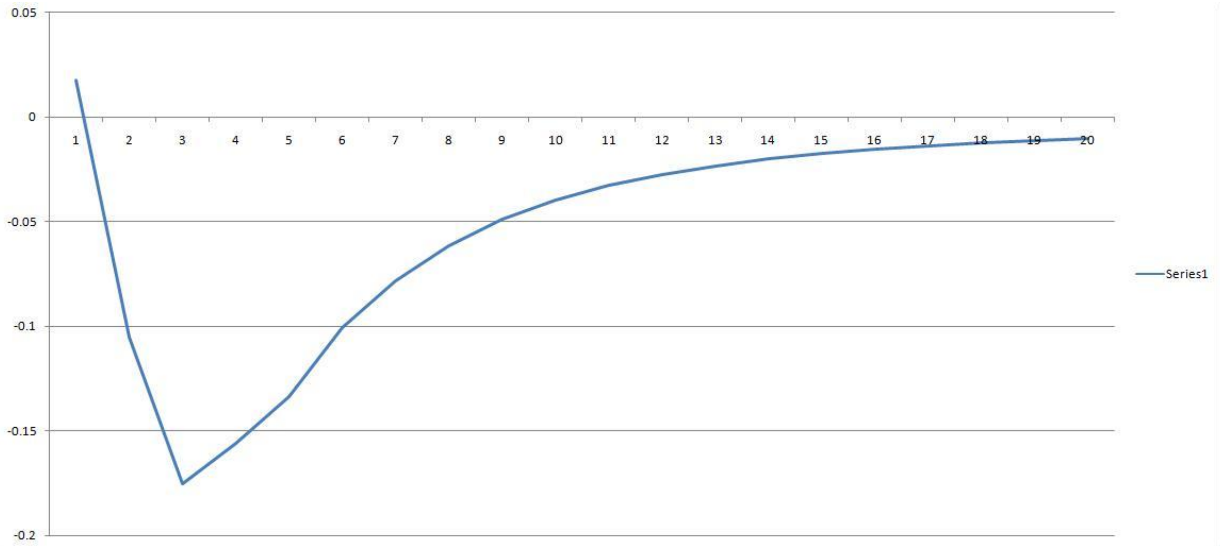

Figure 4: The relation between the values of $\psi^{(2)}$

6. Stability of the Solution

All the previous values $C_i^{(0)}, \beta_i^{(0)}; C_i^{(1)}, \beta_i^{(1)}; a_i^{(1)}, a_i^{(2)}$, the perturbation functions $\psi_i^{(1)}$ and $\psi_i^{(2)}$, $i = 1, 2, 3, \dots, 20$ are explained in the following tables:

Table 1

Values of $C_i^{(0)}$	Values of $\beta_i^{(0)}$	Values of $a_i^{(1)}$	Perturbation $\psi_i^{(1)}$
$C_1^{(0)} = 1$	$\beta_1^{(0)} = 1$	$a_1^{(1)} = -0.06007209978858$	$\psi_1^{(1)} = -0.07608891200005$
$C_2^{(0)} = -0.70989297156176$	$\beta_2^{(0)} = 0.53679193222686$	$a_2^{(1)} = 0.11714700519861$	$\psi_2^{(1)} = 0.12159708772337$
$C_3^{(0)} = 0.12611930213306$	$\beta_3^{(0)} = 0.36145698256312$	$a_3^{(1)} = -0.01492757514491$	$\psi_3^{(1)} = 0.00400395381861$
$C_4^{(0)} = -0.02017710881577$	$\beta_4^{(0)} = 0.27346897244304$	$a_4^{(1)} = 0.00506392385869$	$\psi_4^{(1)} = 0.01536935269110$
$C_5^{(0)} = -0.00374794566663$	$\beta_5^{(0)} = 0.24348474100221$	$a_5^{(1)} = 0.00177865817907$	$\psi_5^{(1)} = 0.00899167758269$
$C_6^{(0)} = -0.00345900351910$	$\beta_6^{(0)} = 0.17340488265495$	$a_6^{(1)} = 0.00141697434545$	$\psi_6^{(1)} = 0.00662001851387$
$C_7^{(0)} = -0.00248317383399$	$\beta_7^{(0)} = 0.14680468709085$	$a_7^{(1)} = 0.00101494061788$	$\psi_7^{(1)} = 0.00486509505371$
$C_8^{(0)} = -0.00189236658938$	$\beta_8^{(0)} = 0.12726445052046$	$a_8^{(1)} = 0.00076838869944$	$\psi_8^{(1)} = 0.00370231839172$
$C_9^{(0)} = -0.00148828196022$	$\beta_9^{(0)} = 0.11227980180411$	$a_9^{(1)} = 0.00060196913306$	$\psi_9^{(1)} = 0.00289997781850$
$C_{10}^{(0)} = -0.00120046530403$	$\beta_{10}^{(0)} = 0.10043121946860$	$a_{10}^{(1)} = 0.00048429512907$	$\psi_{10}^{(1)} = 0.00232739383683$
$C_{11}^{(0)} = -0.00098833088117$	$\beta_{11}^{(0)} = 0.09083719857804$	$a_{11}^{(1)} = 0.00039799079554$	$\psi_{11}^{(1)} = 0.00190647222791$
$C_{12}^{(0)} = -0.00082753606369$	$\beta_{12}^{(0)} = 0.08291940611664$	$a_{12}^{(1)} = 0.00033280034828$	$\psi_{12}^{(1)} = 0.00158901320637$
$C_{13}^{(0)} = -0.00070276168637$	$\beta_{13}^{(0)} = 0.07628291140302$	$a_{13}^{(1)} = 0.00028234182627$	$\psi_{13}^{(1)} = 0.00134425159932$
$C_{14}^{(0)} = -0.00060398214871$	$\beta_{14}^{(0)} = 0.07064974871292$	$a_{14}^{(1)} = 0.00024247183939$	$\psi_{14}^{(1)} = 0.00115193409112$
$C_{15}^{(0)} = -0.00052441251163$	$\beta_{15}^{(0)} = 0.06582018272549$	$a_{15}^{(1)} = 0.00021040280626$	$\psi_{15}^{(1)} = 0.00099835676384$
$C_{16}^{(0)} = -0.00045932219863$	$\beta_{16}^{(0)} = 0.06164987635076$	$a_{16}^{(1)} = 0.00018419990419$	$\psi_{16}^{(1)} = 0.00087402993921$
$C_{17}^{(0)} = -0.00040531730568$	$\beta_{17}^{(0)} = 0.05803808410616$	$a_{17}^{(1)} = 0.00016247994479$	$\psi_{17}^{(1)} = 0.00077227209164$
$C_{18}^{(0)} = -0.00035987980229$	$\beta_{18}^{(0)} = 0.05492898485503$	$a_{18}^{(1)} = 0.00014421976796$	$\psi_{18}^{(1)} = 0.00068837292615$
$C_{19}^{(0)} = -0.00032102387355$	$\beta_{19}^{(0)} = 0.05235503472720$	$a_{19}^{(1)} = 0.00012861482910$	$\psi_{19}^{(1)} = 0.00061920137525$
$C_{20}^{(0)} = -0.00028683743764$	$\beta_{20}^{(0)} = 0.05115084644601$	$a_{20}^{(1)} = 0.00011487273233$	$\psi_{20}^{(1)} = 0.00056360590344$

Table-2

Values of $C_i^{(1)}$	Values of $\beta_i^{(1)}$	Values of $a_i^{(2)}$	Perturbation $\psi_i^{(2)}$
$C_1^{(1)} = 1$	$\beta_1^{(1)} = 1$	$a_1^{(2)} = -0.11511564372714$	$\psi_1^{(2)} = 0.01785226433423$
$C_2^{(1)} = 0.40194580854144$	$\beta_2^{(1)} = -0.54798722378728$	$a_2^{(2)} = -0.11995928999353$	$\psi_2^{(2)} = -0.10523826530048$
$C_3^{(1)} = 0.22338743793847$	$\beta_3^{(1)} = -0.28415156365565$	$a_3^{(2)} = -0.09066255515664$	$\psi_3^{(2)} = -0.177510063075593$
$C_4^{(1)} = 0.13835180460378$	$\beta_4^{(1)} = -0.18688080097151$	$a_4^{(2)} = -0.05242058816528$	$\psi_4^{(2)} = -0.15585986965058$
$C_5^{(1)} = 0.09478884101164$	$\beta_5^{(1)} = -0.12667811727497$	$a_5^{(2)} = -0.03652112468675$	$\psi_5^{(2)} = -0.13351615573417$
$C_6^{(1)} = 0.05981969672545$	$\beta_6^{(1)} = -0.13492306247043$	$a_6^{(2)} = -0.02287112015627$	$\psi_6^{(2)} = -0.10069697972526$
$C_7^{(1)} = 0.04234766463476$	$\beta_7^{(1)} = -0.11293079567521$	$a_7^{(2)} = -0.01622973923624$	$\psi_7^{(2)} = -0.07786891472533$
$C_8^{(1)} = 0.03146305815442$	$\beta_8^{(1)} = -0.09751260780929$	$a_8^{(2)} = -0.01204557433966$	$\psi_8^{(2)} = -0.06112216315538$
$C_9^{(1)} = 0.02422085547716$	$\beta_9^{(1)} = -0.08593561336422$	$a_9^{(2)} = -0.00926203033400$	$\psi_9^{(2)} = -0.04878946678292$
$C_{10}^{(1)} = 0.01918053639987$	$\beta_{10}^{(1)} = -0.07686878039287$	$a_{10}^{(2)} = -0.00732671215120$	$\psi_{10}^{(2)} = -0.03960658532315$
$C_{11}^{(1)} = 0.01554524726800$	$\beta_{11}^{(1)} = -0.06954822620864$	$a_{11}^{(2)} = -0.00593249908134$	$\psi_{11}^{(2)} = -0.03266924467827$
$C_{12}^{(1)} = 0.01284501491195$	$\beta_{12}^{(1)} = -0.06349549402151$	$a_{12}^{(2)} = -0.00489808845618$	$\psi_{12}^{(2)} = -0.02734635600470$
$C_{13}^{(1)} = 0.01078926760143$	$\beta_{13}^{(1)} = -0.05839224689624$	$a_{13}^{(2)} = -0.00411142999652$	$\psi_{13}^{(2)} = -0.02320001158716$
$C_{14}^{(1)} = 0.00919130465877$	$\beta_{14}^{(1)} = -0.05401611226672$	$a_{14}^{(2)} = -0.00350058834966$	$\psi_{14}^{(2)} = -0.01992497574948$
$C_{15}^{(1)} = 0.00792724960164$	$\beta_{15}^{(1)} = -0.05020433588557$	$a_{15}^{(2)} = -0.00301788352575$	$\psi_{15}^{(2)} = -0.01730667358922$
$C_{16}^{(1)} = 0.00691274040421$	$\beta_{16}^{(1)} = -0.04683068188129$	$a_{16}^{(2)} = -0.00263088465836$	$\psi_{16}^{(2)} = -0.01519319461910$
$C_{17}^{(1)} = 0.00608929881074$	$\beta_{17}^{(1)} = -0.04378695608876$	$a_{17}^{(2)} = -0.00231714975701$	$\psi_{17}^{(2)} = -0.01347737666945$
$C_{18}^{(1)} = 0.00541644847608$	$\beta_{18}^{(1)} = -0.04095789945751$	$a_{18}^{(2)} = -0.00206119713152$	$\psi_{18}^{(2)} = -0.01208693000586$
$C_{19}^{(1)} = 0.00486828861100$	$\beta_{19}^{(1)} = -0.03814440559095$	$a_{19}^{(2)} = -0.00185322883452$	$\psi_{19}^{(2)} = -0.01098486164255$
$C_{20}^{(1)} = 0.00443877008410$	$\beta_{20}^{(1)} = -0.03446202188458$	$a_{20}^{(2)} = -0.00169131565626$	$\psi_{20}^{(2)} = -0.01020437007111$

From these tables, we can deduce that, firstly the perturbation in the eigenvalues and eigenfunctions is clearly takes place. Secondly in the middle of the table, we can observe their stability.

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