

Collatz Function like Integral Value Transformations

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Abstract

A class of transformations named as Integral Value Transformation (IVT) on \mathbb{N}_0 , where \mathbb{N}_0 denotes the set of natural numbers along with zero. is introduced. Using IVT an iterative scheme is defined. One of the famous mathematical conjectures in discrete dynamical system is Collatz Conjecture. In this paper, a Collatz function like transformation $IVT^{2,1}_1$ is introduced and we have explored the beauty like the Collatz function. We have proved that the iterative scheme on using the $IVT^{2,1}_1$ converges to zero where as iterative scheme on Collatz function is still a conjecture till today.

Key words: Integral Value Transformation (IVT), Collatz Conjecture, Cellular Automata, IVT-Collatz graph, Distribute Computing System.

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1. Introduction

In modern mathematics, one of the interesting and most enigmatic unsolved mathematical problems is Collatz Conjecture [1, 2] in number theory and discrete dynamical systems proposed by L. Collatz in 1937. Although the problem on which the conjecture is built is remarkably simple to explain and understand, the nature of the conjecture and the behavior of this dynamical system for proving or disproving the conjecture have been altogether exceedingly difficult. In this paper, a class of functions is defined on \mathbb{N}_0^k . One of the functions has been highlighted and demonstrated the Collatz problem like behavior. In fact, there can be a set of other such Collatz like functions in our paradigm. Also one of its applications on distributed and parallel computing has been formalized in this paper.

First of all let us mention the famous problem of ‘Collatz Conjecture’. Let a function T , defined on \mathbb{N} to \mathbb{N} , be defined as follows:

$$T(n) = 3n + 1; \text{ if } n \text{ is odd, } T(n) = \frac{n}{2}; \text{ if } n \text{ is even}$$

Also an iterative scheme is also introduced as $X_{n+1} = T(X_n)$. There is a natural number i such that $X_i = 1$ for all initial value X_0 . This is what is known as Collatz Conjecture [3]. In this context we formulate a similar scenario as the following:

We have defined four IVTs’ in \mathbb{N}_0 . One of those functions, $IVT^{2,1}_1(x) = (2^s - 1) - x$ (where x is an s -bit binary number, which is explained in the next section) behaves like Collatz function. If we consider the iterative scheme like $X_{n+1} = IVT^{2,1}_1(X_n)$ then the iterative sequence can be proved to be converging to 0 for all initial X_0 . The following distinguishing features of our Collatz like sequence are noted below:

- (I) We know that the iterative sequence for Collatz function converges to 1 (with cycle 4, 2, 1) which has been proposed as conjecture for any initial X_0 but this converging point 1 is very much dependent on the initial X_0 in the sense that the ‘i’ could be significantly high depending upon X_0 [4, 5]. On the contrary, in our formulation this ‘i’ can be shown to be very much independent of X_0 and significantly low.
- (II) The converging cycle in Collatz iterative scheme is 4, 2, 1, that is the only way to reach at 1 is reaching the preceding number 2 where as there are several preceding numbers of the form (Merseene number, $2^n - 1$; n is natural number) to come up at 0 in our iterative scheme.

2. Integral Value Transformations

Integral Value Transformations (IVT) $IVT^{p,k}_{\#}$ from \mathbb{N}_0^k to \mathbb{N}_0^k is defined where p denotes the p-adic number, k denotes dimension of the domain and # represents the transformation index [6]. It is worth noting that these IVTs’ correspond to each of the multistate Cellular automata.

Let us define the IVT in \mathbb{N}_0 in 2-adic number systems. There are 4 (2^{2^1}) one variable two state cellular automata rules. These are as follows:

| Variable | f_0 | f_1 | f_2 | f_3 |
|----------|-------|-------|-------|-------|
| 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |

$IVT^{2,1}_{\#}$ mapping a non-negative integers to a non-negative integers.

$$IVT^{2,1}_0(a) = ((f_0(a_n)f_0(a_{n-1}) \dots f_0(a_1))_2 = b$$

$$IVT^{2,1}_1(a) = ((f_1(a_n)f_1(a_{n-1}) \dots f_1(a_1))_2 = b$$

$$IVT^{2,1}_2(a) = ((f_2(a_n)f_2(a_{n-1}) \dots f_2(a_1))_2 = b$$

$$\text{and } IVT^{2,1}_3(a) = ((f_3(a_n)f_3(a_{n-1}) \dots f_3(a_1))_2 = b,$$

where ‘a’ is a non-negative integer and $a = (a_n a_{n-1} \dots a_1)_2$ and ‘b’ is the decimal value corresponding to the binary number.

2.1 Algebraic Form and Their Formulations

The algebraic form of the above transformations is as follows:

$$IVT^{2,1}_0(x) = 0 \text{ for all nonnegative integers.}$$

$$IVT^{2,1}_1(x) = (2^s - 1) - x \text{ where } x \text{ is an } s\text{-bit number representation.}$$

$$IVT^{2,1}_2(x) = x ; \text{ for all non-negative integer } x.$$

$$IVT^{2,1}_3(x) = 2^s - 1 ; x \text{ is an } s\text{-bit number in binary representation.}$$

Now

$$IVT^{2,1}_1(x) = (2^s - 1) - x$$

$$IVT^{2,1}_1(x) = (2^s - 1) - IVT^{2,1}_2(x)$$

$$IVT^{2,1}_2(x) = IVT^{2,1}_3(x) - IVT^{2,1}_1(x)$$

i.e. $IVT^{2,1}_3(x) = IVT^{2,1}_1(x) + IVT^{2,1}_2(x)$ for all non-negative integers x .

Therefore, the relation becomes $IVT^{2,1}_3 = IVT^{2,1}_1 + IVT^{2,1}_2$

2.2 Dynamics of IVTs:

Let us consider an iterative scheme as $X_{n+1} = f(X_n)$ where f is a function from \mathbb{N}_0 to \mathbb{N}_0 . Let us investigate the dynamics of the above four IVTs'.

If we consider f as three IVTs' then we have $X_{n+1} = 0, X_0, 2^s - 1$ for all X_0 (s -bit in binary representation) corresponding to $IVT^{2,1}_0, IVT^{2,1}_2$ and $IVT^{2,1}_3$ respectively. So, iterative scheme for these three transformations are basically static systems. But the transformation $IVT^{2,1}_1$ shows a significant dynamism in the iterative sequence. Let us take one example, let X_0 be 19. 19 (= 10011_2) is a five (s)-bit number. Therefore 19 maps to $(2^5 - 1) - 19 = 12$. A list is shown below in this regard:

Table-1 shows the iterative sequence under $IVT^{2,1}_1$ for given X_0 up to 20.

| X_0 | Iterative sequences |
|-------|---------------------|
| 0 | 0 |
| 1 | 0 |
| 2 | 1, 0 |
| 3 | 0 |
| 4 | 3, 0 |
| 5 | 2, 1, 0 |
| 6 | 1, 0 |
| 7 | 0 |
| 8 | 7, 0 |
| 9 | 6, 1, 0 |
| 10 | 5, 2, 1, 0 |
| 11 | 4, 3, 0 |
| 12 | 3, 0 |
| 13 | 2, 1, 0 |
| 14 | 1, 0 |
| 15 | 0 |
| 16 | 15, 0 |
| 17 | 14, 1, 0 |
| 18 | 13, 2, 1, 0 |
| 19 | 12, 3, 0 |
| 20 | 11, 4, 3, 0 |

Lemma-2. 2. 1

(I) For any non-negative integer of the form $X_0 = 2^n + P$,

$$IVT^{2,1}_1(X_0) = 2^n - (P + 1) \text{ for some non-negative integer.}$$

(II) For any non-negative integer of Merseene form $X_0 = 2^n - 1$

$$IVT^{2,1}_1(X_0) = 0.$$

The proof of the lemma is straightforward.

Theorem-2.2.2

The iterative scheme $\{X_n\}$ converges to 0 for any given X_0 where $X_{n+1} = IVT^{2,1}_1(X_n)$

Proof:

We use Strong Mathematical Induction (SMI) principle to prove the theorem. Let us consider a set T_n , a set of X_0 s which is defined as $\{2^n + p : 0 \leq p \leq 2^n - 1\}$. Clearly for $n = 0, T_0 = \{1\}$, for $n = 1, T_1 = \{2, 3\}$, for $n = 2, T_2 = \{4, 5, 6, 7\}$ and so on...

In this way, all natural numbers along with zero could be captured by the said scheme. Let us define T_1^n as a set of $IVT^{2,1}_1(X_0)$ s corresponding to n for T .

For $n = 0, T_0 = \{1\}$ and so readily $T_1^0 = \{0\}$ i.e. X_0 converges to 0 by Lemma-2.2.1 (II).

For $n = 1$, by lemma-2.2.1 (I) and (II) $T_1 = \{2, 3\}$ becomes $T_1^1 = \{1, 0\} = T_0 \cup T_1^0$. Already, T_0 and T_1^0 have converged to 0.

For $n = 2$, by the lemma 2.2.1 (I) and (II) in $T_2 = \{4, 5, 6, 7\}$ becomes $T_1^2 = \{3, 2, 1, 0\} = T_1 \cup T_1^1$. Previously, T_1 and T_1^1 have converged to 0.

Let us hypothesize that the theorem be true for all $n = m$.

Let us try to establish the theorem is true for $n = m + 1$ also.

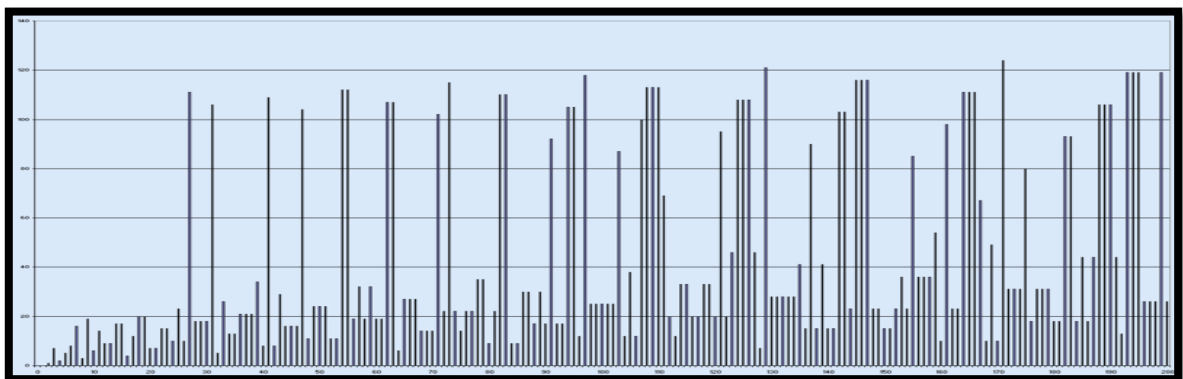
Now, $T_1^{m+1} = \{2^{m+1} + p : 0 \leq p \leq 2^{m+1} - 1\}$ then $T_1^{m+1} = \{p : 0 \leq p \leq 2^{m+1} - 1\} = T_m \cup T_1^m$. According to the SMI hypothesis we could say the iterative scheme is converging to 0.

Therefore the required theorem is followed. (Proved).

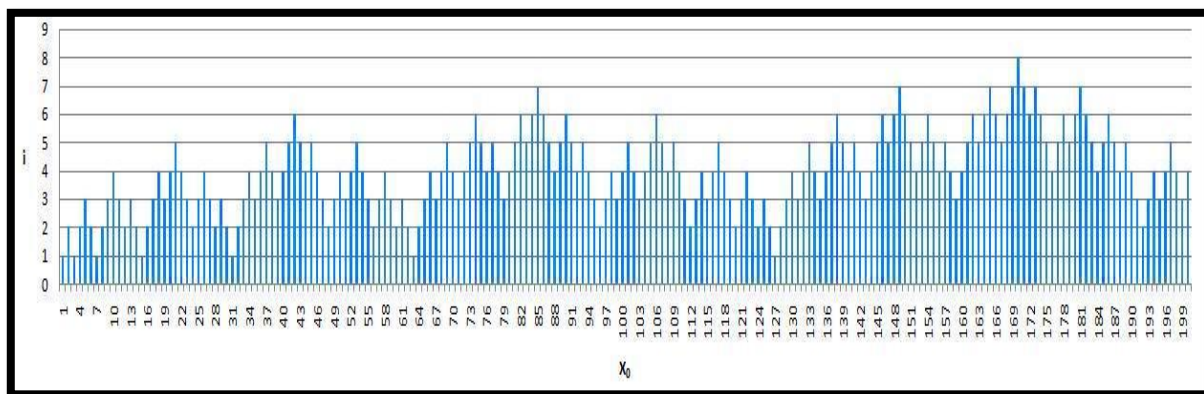
In the subsequent section we would explore the convergence behavior of the iterative scheme corresponding to $IVT^{2,1}_1$ and Collatz function.

2.3 Convergent Dynamism of IVTs and Collatz Function:

Here we are about to enlighten the convergence behavior of Collatz function and $IVT^{2,1}_1$ function. The convergence gateway of Collatz Iterative Scheme (CIS) is $\{4, 2, 1\}$ whereas all Merseene numbers are the gateways for the convergence of $IVT^{2,1}_1$ Iterative Scheme (IIS).



[Figure 1: Collatz Graph (1-200)]

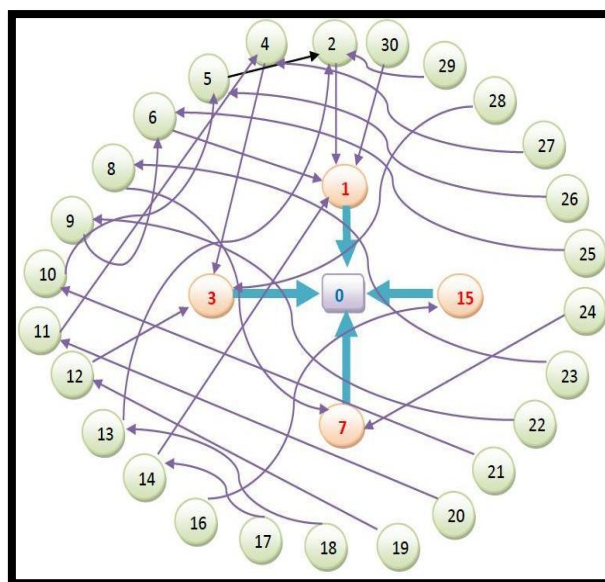


[Figure 2: $IVT^{2,1}_1$ Graph (1-200)]

These Merseene numbers gateways enable the IIS in rapid convergence. This is why the Collatz graph [7] is more chaotic than the $IVT^{2,1}_1$ graph as demonstrated in the above figures.

3. Application on Distributed and Parallel Computing

In our wide spread day to day applications which cover wide geographic area, we may conceive one simple distributed computing scenario for better supervision and control in the Distributed Computing System(DCS) [8] as described in the following paragraph. This supervision interconnection is in addition to the existing distributed computing environment. The convergent point of $IVT^{2,1}_1$ Iterative Scheme (IIS), zero is placed as the *Super Controlling Agent* (SCA) of the whole DCS. This SCA is surrounded by various nodes which may be called Associate SCA (ASCA) designated by the Merseene numbers. And all the existing nodes in the DCS may be designated as arbitrary non-negative integers.



[Figure-3: Proposed Supervision interconnection in the DCS]

For example, for a DCS with 31 number of nodes where 0 plays as the SCA and the nodes 1, 3, 7 and 15 are ASCA and all the rest are common nodes as shown in the figure-3.

So any information transfer when takes place between nodes, from the sender node it reaches to the SCA via the path as shown in figure 3. Therefore, the overall supervision and control can be maintained using the proposed path network which can be run in parallel to the conventional DCS.

In this paper, we have explored the Collatz function like behavior of $IVT^{2,1}_1$ transformation. It is worth noting that the $IVT^{2,1}_1$ is corresponding to p which is 2. So, in this context we would have so many such transformations which are essentially as rich as Collatz function for different p values like 3, 4 and so on. In near future, our research endeavors would be focused on this topic.

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