

DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS IN PHYSICS AND ENGINEERING

PART 1: SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

Historically, partial differential equations originated from the study of surfaces in geometry and a wide variety of problems in mechanics. During the second half of the nineteenth century, many famous mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations. The primary reason for this research was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering.

The next phase of the development of linear partial differential equations was characterized by efforts to develop the general theory and various methods of solution of linear equations. In fact, partial differential equations have been found to be essential to the theory of surfaces on the one hand and to the solution of physical problems on the other. These two areas of mathematics can be seen as linked by the bridge of the calculus of variations. With the discovery of the basic concepts and properties of distributions, the modern theory of linear partial differential equations is now well established. The subject plays a central role in modern mathematics, especially in physics, geometry, and analysis. Almost all physical phenomena obey mathematical laws that can be formulated by differential equations. This striking fact was first discovered by Isaac Newton (1642–1727) when he formulated the laws of mechanics and applied them to describe the motion of the planets [1]. During the three centuries since Newton's fundamental discoveries, many partial differential equations that govern physical, chemical, and biological phenomena have been found and successfully solved by numerous methods. These equations include Euler's equations for the dynamics of rigid bodies and for the motion of an ideal fluid, Lagrange's equations of motion, Hamilton's equations of motion in analytical mechanics, Fourier's equation for the diffusion of heat, Cauchy's equation of motion and Navier's equation of motion in elasticity, the Navier–Stokes equations for the motion of viscous fluids, the Cauchy–Riemann equations in complex function theory, the Cauchy–Green equations for the static and dynamic behavior of elastic solids, Kirchhoff's equations for electrical circuits, Maxwell's equations for electromagnetic fields, and the Schrödinger equation and the Dirac equation in quantum mechanics. This is only a sampling, and the recent mathematical and scientific literature reveals an almost unlimited number of differential equations that have been discovered to model physical, chemical and biological systems and processes [1].

The theory of partial differential equations (PDE) has long been one of the most important fields in mathematics. This is essentially due to the frequent occurrence and the wide range of applications of partial differential equations in many branches of physics, engineering, and other sciences. With much interest and great demand for theory and applications in diverse areas of science and engineering, several works on the applications of PDEs have been published.

In its early stages of development, the theory of second-order linear PDEs was concentrated on applications to mechanics and physics. All such equations can be classified into three basic categories: the wave equation, the heat equation, and the Laplace equation (or potential equation). Thus, a study of these three different kinds of equations yields much

information about more general second-order linear PDEs. Jean d'Alembert (1717–1783) first derived the one-dimensional wave equation for vibration of an elastic string and solved this equation in 1746 [2]. His solution is now known as the d'Alembert solution. The wave equation is one of the oldest equations in mathematical physics. Some form of this equation, or its various generalizations, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. In fact, the studies of water waves, acoustic waves, elastic waves in solids, and electromagnetic waves are all based on this equation. A technique known as the method of separation of variables is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation. The wave equation and its methods of solution attracted the attention of many famous mathematicians including Leonhard Euler (1707–1783), James Bernoulli (1667–1748), Daniel Bernoulli (1700–1782), J. L. Lagrange (1736–1813), and Jacques Hadamard (1865–1963). They discovered solutions in several different forms, and the merit of their solutions and relations among these solutions were argued in a series of papers extending over more than twenty-five years; most concerned the nature of the kinds of functions that can be represented by trigonometric (or Fourier) series. These controversial problems were finally resolved during the nineteenth century [2].

Since there is no time dependence in any of the mathematical problems stated above, there are no initial data to be satisfied by the solutions of the Laplace equation. They must, however, satisfy certain boundary conditions on the boundary curve or surface of a region in which the Laplace equation is to be solved. The problem of finding a solution of Laplace's equation that takes on the given boundary values is known as the Dirichlet boundary-value problem, after Peter Gustav Lejeune Dirichlet (1805–1859). On the other hand, if the values of the normal derivative are prescribed on the boundary, the problem is known as Neumann boundary-value problem, in honor of Karl Gottfried Neumann (1832–1925). Despite great efforts by many mathematicians including Gaspard Monge (1746–1818), Adrien-Marie Legendre (1752–1833), Carl Friedrich Gauss (1777–1855), Simeon-Denis Poisson (1781–1840), and Jean Victor Poncelet (1788–1867), very little was known about the general properties of the solutions of Laplace's equation until 1828, when George Green (1793–1841) and Mikhail Ostrogradsky (1801–1861) independently investigated properties of a class of solutions known as harmonic functions [1]. On the other hand, Augustin Cauchy (1789–1857) and Bernhard Riemann (1826–1866) derived a set of first-order partial differential equations, known as the Cauchy–Riemann equations, in their independent work on functions of complex variables. These equations led to the Laplace equation, and functions satisfying this equation in a domain are called harmonic functions in that domain. Both Cauchy and Riemann occupy a special place in the history of mathematics. Riemann made enormous contributions to almost all areas of pure and applied mathematics. His extraordinary achievements stimulated further developments, not only in mathematics, but also in mechanics, physics, and the natural sciences as a whole.

Historically, Euler first solved the eigenvalue problem when he developed a simple mathematical model for describing the 'buckling' modes of a vertical elastic beam. The general theory of eigenvalue problems for second-order differential equations, now known as the Sturm–Liouville Theory, originated from the study of a class of boundary-value problems due to Charles Sturm (1803–1855) and Joseph Liouville (1809–1882) [3]. They

showed that, in general, there is an infinite set of eigenvalues satisfying the given equation and the associated boundary conditions, and that these eigenvalues increase to infinity. Corresponding to these eigenvalues, there is an infinite set of orthogonal eigenfunctions so that the linear superposition principle can be applied to find the convergent infinite series solution of the given problem. Indeed, the Sturm–Liouville theory is a natural generalization of the theory of Fourier series that greatly extends the scope of the method of separation of variables [1,2]. In 1926, the WKB approximation method was developed by Gregor Wentzel, Hendrik Kramers and Marcel-Louis Brillouin for finding the approximate eigenvalues and eigenfunctions of the one-dimensional Schrödinger equation in quantum mechanics. This method is now known as the short-wave approximation or the geometrical optics approximation in wave propagation theory.

At the end of the seventeenth century, many important questions and problems in geometry and mechanics involved minimizing or maximizing of certain integrals for two reasons. The first of these were several existence problems, such as, Newton’s problem of missile of least resistance, Bernoulli’s isoperimetric problem, Bernoulli’s problem of the brachistochrone (brachistos means shortest, Chronos means time), the problem of minimal surfaces due to Joseph Plateau (1801–1883), and Fermat’s principle of least time. Indeed, the variational principle as applied to the propagation and reflection of light in a medium was first enunciated in 1662 by one of the greatest mathematicians of the seventeenth century, Pierre Fermat (1601–1665). According to his principle, a ray of light travels in a homogeneous medium from one point to another along a path in a minimum time. The second reason is somewhat philosophical, that is, how to discover a minimizing principle in nature. The following 1744 statement of Euler is characteristic of the philosophical origin of what is known as the principle of least action: “As the construction of the universe is the most perfect possible, being the handiwork of all-wise Maker, nothing can be met with in the world in which some maximal or minimal property is not displayed. There is, consequently, no doubt but all the effects of the world can be derived by the method of maxima and minima from their final causes as well as from their efficient ones.” In the middle of the eighteenth century, Pierre de Maupertius (1698–1759) stated a fundamental principle, known as the principle of least action, as a guide to the nature of the universe. A still more precise and general formulation of Maupertius’ principle of least action was given by Lagrange in his *Analytical Mechanics* published in 1788.

The work of Lagrange remained unchanged for about half a century until William R. Hamilton (1805–1865) published his research on the general method in analytical dynamics which gave a new and very appealing form to the Lagrange equations. Hamilton’s work also included his own variational principle. In his work on optics during 1834–1835, Hamilton elaborated a new principle of mechanics, known as Hamilton’s principle, describing the stationary action for a conservative dynamical system.

Indeed, the discovery of the calculus of variations in a modern sense began with the independent work of Euler and Lagrange. The first necessary condition for the existence of an extremum of a functional in a domain leads to the celebrated Euler–Lagrange equation. This equation in its various forms now assumes primary importance, and more emphasis is given to the first variation, mainly due to its power to produce significant equations, than to the second variation, which is of fundamental importance in answering

the question of whether or not an extremal actually provides a minimum (or a maximum). Thus, the fundamental concepts of the calculus of variations were developed in the eighteenth century in order to obtain the differential equations of applied mathematics and mathematical physics. During its early development, the problems of the calculus of variations were reduced to questions of the existence of differential equations problems until David Hilbert developed a new method in which the existence of a minimizing function was established directly as the limit of a sequence of approximations.

With the rapid development of the theory and applications of differential equations, the closed form analytical solutions of many different types of equations were hardly possible. However, it is extremely important and absolutely necessary to provide some insight into the qualitative and quantitative nature of solutions subject to initial and boundary conditions. This insight usually takes the form of numerical and graphical representatives of the solutions. It was E. Picard (1856–1941) who first developed the method of successive approximations for the solutions of differential equations in most general form and later made it an essential part of his treatment of differential equations in the second volume of his *Traité d'Analyse* published in 1896. During the last two centuries, the calculus of finite differences in various forms played a significant role in finding the numerical solutions of differential equations. Historically, many well-known integration formulas and numerical methods including the Euler–Maclaurin formula, Gregory integration formula, the Gregory–Newton formula, Simpson's rule, Adam–Bashforth's method, the Jacobi iteration, the Gauss–Seidel method, and the Runge–Kutta method have been developed and then generalized in various forms [1-3].

With the development of modern calculators and high-speed electronic computers, there has been an increasing trend in research toward the numerical solution of ordinary and partial differential equations during the twentieth century. Special attention has also given to in depth studies of convergence, stability, error analysis, and accuracy of numerical solutions. Many well-known numerical methods including the Crank–Nicolson methods, the Lax–Wendroff method, Richtmyer's method, and Stone's implicit iterative technique have been developed in the second half of the twentieth century. All finite difference methods reduce differential equations to discrete forms. In recent years, more modern and powerful computational methods such as the finite element method and the boundary element method have been developed in order to handle curved or irregularly shaped domains. These methods are distinguished by their more general character, which makes them more capable of dealing with complex geometries, allows them to use non-structured grid systems, and allows more natural imposition of the boundary conditions.

During the second half of the nineteenth century, considerable attention was given to problems concerning the existence, uniqueness, and stability of solutions of partial differential equations. These studies involved not only the Laplace equation, but the wave and diffusion equations as well, and were eventually extended to partial differential equations with variable coefficients. Through the years, tremendous progress has been made on the general theory of ordinary and partial differential equations. With the advent of new ideas and methods, new results and applications, both analytical and numerical studies are continually being added to this subject. Partial differential equations have been the subject of vigorous mathematical research for over three centuries and remain so today.

This is an active area of research for mathematicians and scientists. In part, this is motivated by the large number of problems in partial differential equations that mathematicians, scientists, and engineers are faced with that are seemingly intractable. Many of these equations are nonlinear and come from such areas of applications as fluid mechanics, plasma physics, nonlinear optics, solid mechanics, biomathematics, and quantum field theory. Owing to the ever-increasing need in mathematics, science, and engineering to solve more and more complicated real-world problems, it seems quite likely that partial differential equations will remain a major area of research for many years to come.

Over the second half of the 20th century numerical analysis of partial differential equations has undergone unprecedented development. At its practical end, the vigorous growth and steady diversification of the field were stimulated by the demand for accurate and reliable tools for computational modeling in physical sciences and engineering, and by the rapid development of computer hardware and architecture. At the more theoretical end, the analytical insight into the underlying stability and accuracy properties of computational algorithms for PDEs was deepened by building upon recent progress in mathematical analysis and in the theory of PDEs.

To embark on a comprehensive review of the field of numerical analysis of PDEs within this thesis would have been an impossible task. A paper by Thomee [4] reviews the history of numerical analysis of PDEs, starting with the 1928 paper by Courant, Friedrichs and Lewy on the solution of problems of mathematical physics by means of finite differences. This excellent survey takes the reader through the development of finite differences for elliptic problems from the 1930s, and the intense study of finite differences for general initial value problems during the 1950s and 1960s. The formulation of the concept of stability is explored in the Lax equivalence theorem and the Kreiss matrix lemmas. Reference is made to the introduction of the finite element method by structural engineers, and a description is given of the subsequent development and mathematical analysis of the finite element method with piecewise polynomial approximating functions. The penultimate section of Thomee's survey deals with "other classes of approximation methods", and this covers methods such as collocation methods, spectral methods, finite volume methods and boundary integral methods. The final section is devoted to numerical linear algebra for elliptic problems.

The spline collocation methods, the spectral methods and the wavelet methods are well studied by Bialecki and Fairweather [5], Hesthaven and Gottlieb [6] and Dahmen [7]. The work by Bialecki and Fairweather [5] is a comprehensive overview of orthogonal spline collocation from its first appearance to the latest mathematical developments and applications. The emphasis throughout is on problems in two space dimensions. The paper by Hesthaven and Gottlieb [6] presents a review of Fourier and Chebyshev pseudospectral methods for the solution of hyperbolic PDEs. Emphasis is placed on the treatment of boundaries, stability of time discretization, treatment of non-smooth solutions and multidomain techniques. The paper gives a clear view of the advances that have been made over the last decade in solving hyperbolic problems by means of spectral methods, but it shows that many critical issues remain open. The paper by Dahmen reviews the recent rapid growth in the use of wavelet methods for PDEs. The author focuses on the use of adaptivity, where significant successes have recently been achieved. He describes the

potential weaknesses of wavelet methods as well as the perceived strengths, thus giving a balanced view that should encourage the study of wavelet methods.

Aspects of finite element methods and adaptivity are dealt with in the three papers by Cockburn [8], Rannacher [9] and Suri [10]. The paper by Cockburn is concerned with the development and analysis of discontinuous Galerkin (DG) finite element methods for hyperbolic problems. It reviews the key properties of DG methods for nonlinear hyperbolic conservation laws from a novel viewpoint that stems from the observation that hyperbolic conservation laws are normally arrived at via model reduction, by elimination of dissipation terms. Rannacher's paper is a first-rate survey of duality-based a posteriori error estimation and mesh adaptivity for Galerkin finite element approximations of PDEs.

The approach is illustrated for simple examples of linear and nonlinear PDEs, including also an optimal control problem. Several open questions are identified such as the efficient determination of the dual solution, especially in the presence of oscillatory solutions. The paper by Suri is a lucid overview of the relative merits of the hp and p versions of the finite element method over the h version. The work is presented in a non-technical manner by focusing on a class of problems concerned with linear elasticity posed on thin domains. This type of problem is of considerable practical interest and it generates a number of significant theoretical problems.

Iterative methods and multigrid techniques are reviewed in a paper by Silvester, Elman, Kay and Wathen [11], and in three papers by Stiiben [12], Wesseling and Oosterlee [13] and Xu [14]. The paper by Silvester et al. outlines a new class of robust and efficient methods for solving linear algebraic systems that arise in the linearisation and operator splitting of the Navier-Stokes equations. A general preconditioning strategy is described that uses a multigrid F-cycle for the scalar convection-diffusion operator and a multigrid F-cycle for a pressure Poisson operator. This two-stage approach gives rise to a solver that is robust with respect to time-step-variation and for which the convergence rate is independent of the grid. The paper by Stiiben gives a detailed overview of algebraic multigrid. This is a hierarchical and matrix-based approach to the solution of large, sparse, unstructured linear systems of equations. It may be applied to yield efficient solvers for elliptic PDEs discretized on unstructured grids. The author shows why this is likely to be an active and exciting area of research for several years in the new millennium. The paper by Wesseling and Oosterlee reviews geometric multigrid methods, with emphasis on applications in computational fluid dynamics (CFD). The paper is not an introduction to multigrid: it is more appropriately described as a refresher paper for practitioners who have some basic knowledge of multigrid methods and CFD. The authors point out that textbook multigrid efficiency cannot yet be achieved for all CFD problems and that the demands of engineering applications are focusing research in interesting new directions. Semi-coarsening, adaptivity and generalization to unstructured grids are becoming more important. The paper by Xu presents an overview of methods for solving linear algebraic systems based on subspace corrections. The method is motivated by a discussion of the local behavior of high-frequency components in the solution of an elliptic problem. Of novel interest is the demonstration that the method of subspace corrections is closely related to von Neumann's method of alternating projections. This raises the question as to whether certain error estimates for alternating directions that are available in the literature

may be used to derive convergence estimates for multigrid and/or domain decomposition methods.

Moving finite element methods and moving mesh methods are presented, respectively, in the papers by Baines [15] and Huang and Russell [16]. The paper by Baines reviews recent advances in Galerkin and least-squares methods for solving first- and second order PDEs with moving nodes in multi dimensions. The methods use unstructured meshes and they minimize the norm of the residual of the PDE over both the computed solution and the nodal positions. The relationship between the moving finite element method and L_2 least-squares methods is discussed. The paper also describes moving finite volume and discrete l_2 least-squares methods. Huang and Russell review a class of moving mesh algorithms based upon a moving mesh partial differential equation (MMPDE). The authors are leading players in this research area, and the paper is largely a review of their own work in developing viable MMPDEs and efficient solution strategies.

In [17] the authors presented the exact solutions of the partial differential equations in different dimensions with variable coefficients by using the homotopy perturbation method. The feature of this method is its flexibility and ability to solve parabolic-like equations and hyperbolic-like equations without the calculation of complicated Adomian polynomials or unrealistic nonlinear assumptions. The numerical results show that this method is a promising and powerful tool for solving the partial differential equations with variable coefficients.

In [18] a new approach is proposed to solve partial differential equations. This method is based on generalized Taylor's formula. The solution of partial differential equations can be expanded using MAPLE. Doing some simple mathematical operations on these equations, one can get a closed form series solution or approximate solution quickly. PDE problems with constant and variable coefficients are solved by the present method. With this method, one can reach same results simpler way than other analytical or approximate methods.

In [19] the stability of the laminar flow between two rotating cylinders (Taylor–Couette flow) is numerically studied. The simulation is based on the equations of motion of an inviscid fluid (Euler equations). The influence exerted on the flow stability by physical parameters of the problem (such as the gap width between the cylinders, the initial perturbation, and the velocity difference between the cylinders) is analyzed. It is shown that the onset of turbulence is accompanied by the formation of large vortices. The results are analyzed and compared with those of similar studies.

A high-order accurate method for analyzing two-dimensional rarefied gas flows is proposed in [20] based on a nonstationary kinetic equation in arbitrarily shaped regions. The basic idea behind the method is the use of hybrid unstructured meshes in physical space. Special attention is given to the performance of the method in a wide range of Knudsen numbers and to accurate approximations of boundary conditions. Examples calculations are provided.

The operator whose spectrum is studied in [21] corresponds to linearized stationary equations of viscous compressible fluid in R^3 , with periodic boundary conditions. The

equations are obtained by linearization of the nonlinear model equations of viscous compressible fluid near an arbitrary solution depending on the variable \mathbf{x} . It is proved that the operator in question is sectorial and that its spectrum is discrete. Also, a subset of the complex plane that contains the spectrum is described. The resolvent is estimated off a sector in the complex plane that is symmetric with respect to the real axis.

The authors in [22] considered the numerical simulation of the two-dimensional viscous flow over a solid ellipse with an aspect ratio equal 3.5. Sufficiently far from the ellipse, the flow is assured potential. The flow is modeled by the two-dimensional partial differential equations of conservation of mass and moment, using elliptic coordinates. The finite volume method is used to discretize the model equations. The numerical solutions revealed that the flow over the ellipse is steady with zero vortex up to $Re = 40$. For Reynolds numbers between 50 and 190, the flow is steady with two vortices in the wake. For $Re = 210$ the flow becomes unstable with harmonic oscillations. The two vortices are alternate in the time with a Strouhal number equal to 0.2075. For the Reynolds number between 220 and 280 the vortices are detached one after other. The spectral analysis of the discrete time variation of the flow velocity at a point within the upper vortex shows that the dominant oscillations frequency is $f = 0.2748$.

A numerical study of heat transfer enhancement due to the deformation of droplets at high Reynolds numbers is described in [23]. The two phase-flow has been computed with a 3D-DNS program using the volume-of-fluid method. The droplets are deformed because of the surrounding gas stream especially due to a sudden rise of flow velocity from zero to U_i . As the governing non-dimensional parameter, the Weber number of the droplets has been varied between 1.3 and 10.8 by assuming different surface tensions at Reynolds numbers between 360 and 853. The dynamical behavior of the droplets as a function of the Weber and the Ohnsorge number are in good agreement with experimental results from the literature. At the highest Reynolds number $Re = 853$, a significant dependency of Nu on We has been found. The comparison of a Nusselt number computed with the real surface area with a Nusselt number computed with the spherical surface area shows that the heat transfer increases not only due to the droplet motion but also due to the larger surface area of the deformed droplet.

The latest developments in the simulation of turbulence by detached eddy simulation (DES) have suggested that this technique might be able to replace large eddy simulation (LES) within the next decade [24]. The results of the flow past a square cylinder show that this approach is quite inexpensive compared to LES while capturing the most important features of the flow. The study in [24] extends the range of applications of DES towards a fully unsteady three-dimensional case with strong streamline curvature, which is known to be a major problem for Reynolds-averaged Navier–Stokes equation (RANS) methods. The case considered in [24] is the turbulent flow over wall-mounted cubes at a Reynolds number of $Re = 1.3 \times 10^4$. The results demonstrate that DES is able to capture the most dominant flow patterns like LES, while RANS only gives only a poor representation of the unsteady flow phenomena.

The fluid flow and heat transfer from a stationary cube placed in a uniform flow is studied numerically in [25]. The three-dimensional unsteady Navier Stokes and energy

equations are solved using higher order temporal and spatial discretization. Computations are carried out for a Reynolds number range of 50–400. At $Re = 218$, the symmetry seen at $Re = 216$ breaks down in one of the orthogonal planes while remains symmetric on the other thus showing a planar symmetry. The flow experiences a Hopf bifurcation at a Reynolds number between 265 and 270 and becomes unsteady. The thermal field also shows all the transitions same as those of flow transitions. The drag coefficient decreases while the heat transfer shows an increasing trend with Reynolds number. The transition from a steady to an unsteady flow does not show any significant increase in the heat transfer. Both the flow and thermal fields show multiple frequencies at high Reynolds number and the number of frequencies increases with the increase in Reynolds number. The instantaneous flow and temperature field are seen to deviate from planar symmetry at $Re = 400$.

The effect of blockage ratio on the flow characteristics of power-law fluids across a square cylinder confined in a channel has been investigated in [26] for the range of conditions $1 \leq Re \leq 45$, $0.5 \leq n \leq 2.0$ and $\beta = 1/8, 1/6$ and $1/4$. Extensive numerical results on the individual and total drag coefficients, wake length, stream function, vorticity and power-law viscosity on the surface of the square cylinder are reported to determine the combined effects of the flow behavior index, blockage ratio and Reynolds number. The size of the wake region is influenced more by blockage than by power-law index. Similarly, drag is also seen to be more influenced by blockage ratio and the Reynolds number than that by the power-law index.

The two-dimensional flow of power-law fluids over an isolated unconfined square cylinder has been investigated numerically in [27] in the range of conditions $1 \leq Re \leq 45$ and $0.5 \leq n \leq 2.0$. The global quantities such as wake length, drag coefficients and the detailed kinematic variables like stream function, vorticity and so on, have been calculated for the above range of conditions. In particular, the effects of Reynolds number and of the power-law index have been investigated in the steady flow regime. The shear-thinning fluid behavior increases the drag above its Newtonian value whereas the shear-thickening behavior reduces the drag below its Newtonian value. However, as the value of the Reynolds number is gradually increased, the role of power-law index diminishes. Similarly, the wake size is shorter in shear-thinning fluids than that in Newtonian fluids under otherwise identical conditions.

The paper's focus in [28] is the calculation of unsteady incompressible 2D flows past airfoils. In the framework of the primitive variable Navier–Stokes equations, the initial and boundary conditions must be assigned so as to be compatible, to assure the correct prediction of the flow evolution. This requirement, typical of all incompressible flows, viscous or inviscid, is often violated when modeling the flow past immersed bodies impulsively started from rest. Its fulfillment can however be restored by means of a procedure enforcing compatibility, consisting in a pre-processing of the initial velocity field, here described in detail. Numerical solutions for an impulsively started multiple airfoil have been obtained using a finite element incremental projection method. The spatial discretization chosen for the velocity and pressure are of different order to satisfy the inf-sup condition and obtain a smooth pressure field. Results are provided to illustrate the effect of employing or not the compatibility procedure and are found in good agreement

with those obtained with a non-primitive variable solver. In addition, the authors introduce a post-processing procedure to evaluate an alternative pressure field which is found to be more accurate than the one resulting from the projection method. This is achieved by considering an appropriate 'unsplit' version of the momentum equation, where the velocity solution of the projection method is substituted.

Quite effective low-order finite element and finite volume methods for incompressible fluid flows have been established and are widely used. However, higher-order finite element methods that are stable, have high accuracy and are computationally efficient are still sought. Such discretization schemes could be particularly useful to establish error estimates in numerical solutions of fluid flows. The objective of the study in [29] is to report on a study in which the cubic interpolated polynomial (CIP) method is embedded into 4-node and 9-node finite element discretization of 2D flows in order to stabilize the convective terms. To illustrate the capabilities of the formulations, the results obtained in the solution of the driven flow square cavity problem are given.

A numerical study on the uniform shear flow past a long cylinder of square cross-section placed parallel to a plane wall has been made in [30]. The cylinder is within the boundary layer of the wall. The maximum gap between the plane wall to the cylinder is taken to be 0.25 times the cylinder height. The authors investigated the flow when the regular vortex shedding from the cylinder is suppressed. The governing unsteady Navier-Stokes equations are discretized through the finite volume method on staggered grid system. A pressure correction based iterative algorithm, SIMPLER, has been used to compute the discretised equations iteratively. The authors found that the critical value of the gap height for which vortex shedding is suppressed depends on the Reynolds number, which is based on the height of the cylinder and the incident stream at the surface of the cylinder. At high Reynolds number ($Re \geq 500$) however, a single row of negative vortices occurs for wall to cylinder gap height $L \geq 0.2$. The shear layer that emerges from the bottom face of the cylinder reattaches to the cylinder itself at this gap height.

Hydrodynamic equations for ideal incompressible fluid are written in [31] in terms of generalized stream function. Two-dimensional version of these equations is transformed to the form of one dynamic equation for the stream function. This equation contains arbitrary function which is determined by inflow conditions given on the boundary. To determine unique solution, velocity and vorticity (but not only velocity itself) must be given on the boundary. This unexpected circumstance may be interpreted in the sense that the fluid has more degrees of freedom than it was believed. Besides, the vorticity is a less observable quantity as compared with the velocity. It is shown that the Clebsch potentials are used essentially at the description of vortical flow.

On the other hand, four mixed problems for the string vibration equation with boundary conditions and homogeneous nonlocal conditions of the first or second kind and with zero initial conditions have been considered in [32]. Using recursion relations, the author found the generalized solutions of the above-mentioned problems.

In [33] the authors reviewed the research on the vibration of orthotropic membrane, which commonly applied in the membrane structural engineering. They applied the large deflection theory of membrane to derive the governing vibration equations of orthotropic

membrane, solved it, and obtained the power series formula of nonlinear vibration frequency of rectangular membrane with four edges fixed. The paper gave the computational example and compared the two results from the large deflection theory and the small one, respectively. Results obtained from this paper provide some theoretical foundation for the measurement of pretension by frequency method; meanwhile, the results provide some theoretical foundation for the research of nonlinear vibration of membrane structures and the response solving of membrane structures under dynamic loads.

In [34], an analytical approach for free vibration analysis of rectangular and circular membranes is presented. The method is based on wave approach. From wave standpoint vibration propagate, reflect and transmit in a structure. Firstly, the propagation and reflection matrices for rectangular and circular membranes are derived. Then, these matrices are combined to provide a concise and systematic approach to free vibration analysis of membranes. Subsequently, the eigenvalue problem for free vibration of membrane is formulated and the equation of membrane natural frequencies is constructed. Finally, the effectiveness of the approach is shown by comparison of the results with existing classical solution.

Free vibration problems of membrane have been solved by several authors, in the past [35-50]. Buchanan and Peddieson [35,36] and Buchanan [37] used Ritz and finite element method respectively, for vibration analysis of circular and elliptic membranes with variable density. Exact power series solutions for axisymmetric vibrations of circular and annular membranes with continuously varying density were presented by Willatzen [49]. Analytical solutions of the free vibration problems of arbitrarily shaped membranes have been investigated by Kang et al. [44,45] and Kang and Lee [43] using non-dimensional dynamic influence function. Radial basis function-based differential quadrature method was used for free vibration analysis of arbitrary shaped membrane by Wu et al. [50]. The method differential quadrature was applied for frequency analysis of rectangular and circular membranes by Laura et al. [46,47]. Some important studies concerning analysis of membranes have been carried out, namely by Leung et al. [48], Houmat [40,41], Gutierrez et al. [39], Irie et al. [42]. Because of its relationship to the wave equation, the Helmholtz equation arises in problems in such areas of mathematical physics as the study of acoustics. The method of discrete singular convolution [51] has been used recently for the vibration analysis of structures. Discrete singular convolution (DSC) method has emerged as a new approach for numerical solutions of differential equations. This new method has a potential approach for computer realization as a wavelet collocation scheme [52-59] has been proven to be quite satisfactory. Free vibration analysis of plates and shells has also been investigated by the present author [60-65]. The aim of the present paper is to present the DSC method for free vibration analysis of membranes having different geometries.

Free vibration analysis of curvilinear membranes is presented in [66]. Irregular physical domain is transformed into a rectangular domain by using geometric coordinate transformation via an eight-noded element. Some numerical examples are provided on membranes with different geometry such as, sectorial, annular sectorial, and elliptic membranes with four curved edges. The results obtained by the DSC method are compared with those obtained by other numerical and analytical methods. It is shown that reasonable accurate results are obtained.

In [67], Discrete Singular Convolution (DSC) method is developed for free vibration analysis of plates and membranes with trapezoidal shape. The straight-sided quadrilateral domain is mapped into a square domain in the computational space using a four-node element. By using the geometric transformation, the governing equations and boundary conditions of the plate are transformed from the physical domain into a square computational domain. Numerical examples illustrating the accuracy and convergence of the DSC method for trapezoidal plates and membranes are presented. The results obtained by DSC method were compared with those obtained by the other numerical and analytical methods.

In [68] a numerical method of solution of some partial differential equations is presented. The method is based on representation of Green functions of the equations in the form of functional integrals and subsequent approximate calculation of the integrals with the help of a deterministic approach. In this case the solution of the equations is reduced to evaluation of usual (Riemann) integrals of relatively low multiplicity. A procedure allowing one to increase accuracy of the solutions is suggested. The features of the method are investigated on examples of numerical solution of the Schrödinger equation and related diffusion equation.

As an example of the applications of the second order linear parabolic partial differential equations, the authors in [69-72] studied the solutions of the single particle time dependent Schrödinger wave equation for a nucleon (proton or neutron) which is moving in an average field, created due to the presence of the other $(A - 1)$ particles, where A is the number of nucleons in the nucleus (the mass number). They carried out this treatment in framework of the so-called single particle Schrödinger fluid. The single particle Schrödinger fluid [73] is a concept which is used to describe the motion of a single nucleon in an axially deformed potential of the nucleus. This concept is carried out by a suitable choice of the time- dependent part of the nucleon wave function in the time-dependent Schrödinger equation. This concept can be applied to study the rotational motion of a deformed nucleus. Accordingly, the derivations of this concept have been carried out and the moment of inertia of an axially deformed nucleus can be obtained in framework of this concept [69-72].

This work is written to present an approach based mainly on the mathematics, physics, and engineering problems and their solutions, and also to construct three physical projects on the three different branches of second order linear partial differential equations which are of particular interest for researchers. Our primary objective, therefore, is not concerned with an elegant exposition of general theory, but rather to provide the fundamental concepts, the underlying principles, a wide range of applications, and various methods of solutions of second order partial differential equations.

Accordingly, the purpose of the present work is to give an insight study of the second-order partial differential equations which have wide range of applications in theoretical physics and engineering problems. The methods of solutions and the numerical tools applied for these types of PDE are of particular interest in this concern. We have forwarded our attention equally to the three well-known types of the second order PDE, namely: elliptic, hyperbolic and parabolic PDE. Accordingly, we have studied one major problem in each case.

As an application of second-order elliptic PDE in applied mathematics, we studied in details the problem of the flows of fluid passing a rectangular plate, where the boundary conditions are of particular importance in this study. Also, the numerical method of solution is very essential in this study.

Furthermore, as an example on the application of the second-order hyperbolic PDE in engineering problems, we have studied the vibrations of membranes, both rectangular and circular, which are subjected to restorative forces proportional to the velocity. The initial conditions here are of extremely importance in this investigation.

The third example in this work is related to the application of the second-order parabolic PDE in quantum mechanics. Accordingly, we have studied the problem of introducing the concepts of fluid mechanics in quantum mechanics, namely in the so called single-particle Schrödinger fluid. In this concept the constituent particle of the nucleus, nucleon, is assumed to move independently in an average potential, assumed deformed, which represents the action of all the other particles on this nucleon. The governing equation in this study is the single-particle time-dependent Schrödinger equation, which is of a parabolic type.

The first chapter of this work gives an introduction to linear PDEs of second order in two and three independent variables and their classifications into hyperbolic, parabolic, and elliptic types. The methods of solutions of these equations are also given in this chapter.

The second chapter deals with the mathematical models representing physical and engineering problems that yield the three basic types of PDEs. Included are only important equations of most common interest in physics and engineering. The numerical methods of solutions are also discussed in this chapter.

The third chapter constitutes a project from fluid mechanics on the applications of second order linear elliptic PDE in continuous medium, namely; Project-1: “Two-Dimensional Fluid Flow Past a Rectangular Plate”.

Chapter-4 is dedicated to the applications of parabolic PDE in Quantum mechanics, by investigating the methods of solutions of the time-dependent Schrödinger wave equation for a nucleon which is moving in a deformed time-dependent potential in framework of the so-called single particle Schrödinger fluid, namely; Project-2: “Single Particle Schrödinger Fluid and Moments of Inertia of the Even-Even Deformed Nuclei in the sd -shell”.

In Chapter-5 we have dealt with a project on the applications of second order hyperbolic PDE in waves, namely; Project-3: “Perturbation Treatment for the Vibrations of Membranes Subjected to A Restorative Force”.

CHAPTER - 1

LINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR METHODS OF SOLUTIONS

1.1 Types of Equations

As is well known, a physical problem is not uniquely specified if we simply give the differential equation which the solution must satisfy, for there are an infinite number of solutions of every equation that describes the physical problem. In order to make the problem a definite one, with a unique answer, we must pick, out of the mass of possible solutions, the one which has certain definite properties along definite boundary surfaces. Any physical problem must state not only the differential equation which is to be solved but also the boundary conditions which the solution must satisfy. The satisfying of the boundary conditions is often as difficult a task as the solving of the differential equation.

The first fact which we must notice is that we cannot try to make the solutions of a given equation satisfy any sort of boundary conditions; we should not try to “squeeze a right-hand foot into a left-hand shoe,” so to speak. For each type of equation there is a definite set of boundary conditions which will give unique answers, and any other sort of conditions will give non unique or impossible answers. Now, of course, an actual physical problem will always have the right sort of boundary conditions to give it a unique answer (or, at least, so we all hope), and if we make our statement of the problem correspond to the actualities, we shall always have the right boundary conditions for the equations. But it is not always easy to tell just what boundary conditions correspond to “actuality,” and it is well for us to know what conditions are suitable for what equations so we can be guided in making our mathematical problems fit the physical problems as closely as possible.

In order to clarify the types of second order PDEs, let us first discuss a two-dimensional example in order to bring out the concepts without confusing by complexity. All the two-dimensional PDEs for scalar field and many of the equations for components of vector fields have the general form [74]

$$A(x, y) \frac{\partial^2 \psi}{\partial x^2} + 2B(x, y) \frac{\partial^2 \psi}{\partial x \partial y} + C(x, y) \frac{\partial^2 \psi}{\partial y^2} = F\left(x, y, \psi, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right), \quad (1.1)$$

where, if the equation is linear in ψ , F has the form

$$D(x, y) \frac{\partial \psi}{\partial x} + E(x, y) \frac{\partial \psi}{\partial y} + G(x, y) \psi + H(x, y). \quad (1.2)$$

There is a non-denumerable infinity of solutions of this equation; the additional conditions imposed by the problem, which serve to fix on one particular solution as being appropriate, are called boundary conditions. Usually they take the form of the specification of the

behavior of the solution on or near some boundary line (or surface, in three dimensions). (From this point of view, initial conditions are just boundary conditions in time). It naturally is of interest to see what sort of specification of the field along the line there must be in order that a unique answer result.

1.2 Types of Boundary Conditions

In every case of the solutions of a certain PDE we must specify the shape of the boundary. It may be a closed curve for the Laplace equation in two space dimensions, or it may be an open, U-shaped boundary consisting of a line parallel to the space axis and two lines parallel to the time axis for a string (wave equation in time and one space dimension) fixed at the ends and given a specified start at a given time, and so on. The boundary is closed if it completely surrounds the solution (even if part of the boundary is at infinity); it is open if the boundary goes to infinity and no boundary conditions are imposed along the part at infinity [74].

In a one-dimensional case, the solution of a second-order equation is uniquely specified if we specify its initial value and slope. By analogy we might expect that, if the boundary were parallel to one of the axes, specification of the value of ψ along the boundary [i.e., specifying $\psi(s)$] and of the gradient of ψ normal to the boundary [i.e., specifying $N(s)$, in this case $\partial\psi/\partial y$] will uniquely fix the solution. This is correct but it is too special a case to satisfy us. We should sometimes like to have a boundary which is not contiguous with a coordinate line but is free to take any reasonable shape. It is not quite so obvious that specification of value and normal gradient on a boundary of any shape will give a unique result (nor is it true), and we shall have to determine the sort of boundaries which are satisfactory.

The type of boundary condition mentioned in the last paragraph, the specifying of value and normal gradient, is called the Cauchy boundary condition [74], and the problem of determining the shape of boundary and type of equation which yields unique and reasonable solutions for Cauchy conditions is called the Cauchy problem, after the investigator who first studied it in detail. Specifying the initial shape and velocity of an infinite flexible string corresponds to Cauchy conditions along the line $t = \text{constant}$. As we know, this uniquely specifies the solution.

On the other hand, if the solution is to be set up inside a closed boundary, it might be expected that Cauchy conditions are too much requirement and might rule out all solutions. Perhaps one only needs to specify the value $\psi(s)$ alone or the normal gradient $N(s)$ alone along the boundary in order to obtain a unique answer.

The specifying only of values along the boundary is called Dirichlet conditions, and the specifying only of slopes is called Neumann conditions. A potential problem, such as the determination of electric potential inside a sequence of conductors at specified potentials, corresponds to Dirichlet conditions. On the other hand, the determination of velocity potential around solid bodies, where the fluid must flow tangential to the surface of the solids and the normal potential gradient at the surface is zero, corresponds to Neumann

conditions [74]. Alternatively, we may, at times, need to specify the value of some linear combination of value and slope, a single boundary condition which is intermediate between Dirichlet and Neumann conditions.

In terms of our supporting line in Figure-1.1, Cauchy conditions correspond to our specifying not only the line $\psi(s) = z$, but also the normal slope at the edge of the surface $\psi(x, y) = z$. It is as though, instead of a line, we had a thin ribbon as a support to the ψ surface, a twisted ribbon which specified slope perpendicular to its axis as well as height above the z axis (but not higher derivatives). For Dirichlet conditions the supporting line is really a line, not a ribbon [74]. For Neumann conditions the ribbon is free to move up and down, only the "slant" of the ribbon is fixed. Sometimes these two conditions are homogeneous, when $\alpha\psi(s) + \beta N(s) = 0$, for α, β specified but independent of s , and sometimes the conditions are inhomogeneous, when $\alpha\psi(s) + \beta N(s) = F(s)$.

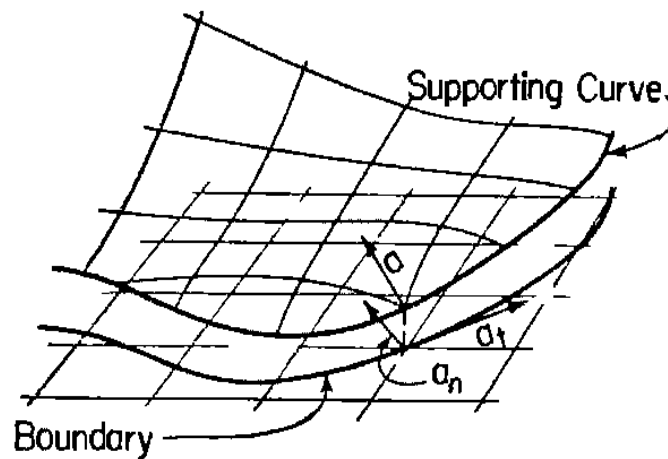


Figure-1.1 boundary conditions in two-dimensions. Surface $z = \psi(x, y)$ boundary curve $x = \xi(s)$, $y = \eta(s)$, unit vectors \mathbf{a}_t , and \mathbf{a}_n in x, y plane; vector \mathbf{a} tangent to surface at boundary.

But we now must go back to our general equation (1.1) and see under what circumstances Cauchy conditions along the curve $= \xi(s)$, $y = \eta(s)$ will result in a unique solution.

In the following, we give a note on the Cauchy problem and the characteristic curve by computing the function ψ . In order to compute ψ at some distance away from the boundary we can have recourse to some two-dimensional power series, analogous to Taylor's series:

$$\psi(x, y) = \psi(\xi, \eta) + \left[(x - \xi) \frac{\partial \psi}{\partial x} + (y - \eta) \frac{\partial \psi}{\partial y} \right] + \frac{1}{2} \left[(x - \xi)^2 \frac{\partial^2 \psi}{\partial x^2} + 2(x - \xi)(y - \eta) \frac{\partial^2 \psi}{\partial x \partial y} + (y - \eta)^2 \frac{\partial^2 \psi}{\partial y^2} \right] + \dots \quad (1.3)$$

where ψ and all its derivatives on the right-hand side of the equation are evaluated at the boundary point (ξ, η) . Once these partial derivatives of ψ are all evaluated at the boundary,

then ψ is uniquely specified within the radius of convergence of the series, *i.e.*, over all of a strip contiguous to the boundary line, which may be infinite in width depending on the nature of the equation. If we can work out a recipe for computing the partial derivatives, we shall have the Cauchy problem well along toward solution [74]. This is not so straightforward as it may at first seem, for we are given only the equation for ψ , the parametric equations for the boundary, and the values of $\psi(s)$ and $N(s)$ on the boundary and from these data are to compute all the double infinity of values of the partial derivatives for each point (ξ, η) on the boundary.

It is not too difficult to express the first derivatives in terms of known quantities. There are two of them, and there are two equations, one giving the specified normal gradient $N(s)$ and the other the rate of change of the known value $\psi(s)$ along the boundary:

$$N(s) = \left(\frac{d\eta}{ds}\right) \left(\frac{\partial\psi}{\partial x}\right) - \left(\frac{d\xi}{ds}\right) \left(\frac{\partial\psi}{\partial y}\right) = a_n \cdot \text{grad } \psi ; \text{ at } x = \xi, y = \eta$$

$$\frac{d}{ds} \psi(s) = \left(\frac{d\xi}{ds}\right) \left(\frac{\partial\psi}{\partial x}\right) + \left(\frac{d\eta}{ds}\right) \left(\frac{\partial\psi}{\partial y}\right) = a_t \cdot \text{grad } \psi ; \text{ at } x = \xi, y = \eta$$

Since the determinant of the coefficients $(d\xi/ds)^2 + (d\eta/ds)^2 = 1$, there is always a solution for these equations:

$$\left(\frac{\partial\psi}{\partial x}\right)_{\xi, \eta} = N(s) \left(\frac{d\eta}{ds}\right) + \left(\frac{d\xi}{ds}\right) \left(\frac{d\psi}{ds}\right) = p(s) ;$$

$$\left(\frac{\partial\psi}{\partial y}\right)_{\xi, \eta} = \left(\frac{d\eta}{ds}\right) \left(\frac{d\psi}{ds}\right) - \left(\frac{d\xi}{ds}\right) N(s) = q(s) \quad (1.4)$$

But the next step, to obtain the second derivatives, is not so simple. It is also the crucial step, for if we can find the three second partials, we shall find that solving for the higher derivatives is simply "more of the same." Now that we have solved for the first derivatives, we know p and q , given in Eq. (1.4) as functions of the parameter s . Two of the needed three equations for the second derivatives are obtained by writing down the expression for the known rate of change of p and q with s in terms of these second derivatives; the third equation is the differential equation ψ must satisfy, Eq. (1.1) itself:

$$\left(\frac{d\xi}{ds}\right) \left(\frac{\partial^2\psi}{\partial x^2}\right) + \left(\frac{d\eta}{ds}\right) \left(\frac{\partial^2\psi}{\partial x\partial y}\right) = \frac{dp}{ds}$$

$$\left(\frac{d\xi}{ds}\right) \left(\frac{\partial^2\psi}{\partial x\partial y}\right) + \left(\frac{d\eta}{ds}\right) \left(\frac{\partial^2\psi}{\partial y^2}\right) = \frac{dq}{ds}$$

$$A(s) \left(\frac{\partial^2\psi}{\partial x^2}\right) + 2B(s) \left(\frac{\partial^2\psi}{\partial x\partial y}\right) + C(s) \left(\frac{\partial^2\psi}{\partial y^2}\right) = F(s)$$

where $A(s)$, etc., are the known values of the coefficients at the point $\xi(s)$, $\eta(s)$ on the boundary.

These three equations can be solved, to find the three partials, unless the determinant of the coefficients

$$\Delta = \begin{vmatrix} \frac{d\xi}{ds} & \frac{d\eta}{ds} & 0 \\ 0 & \frac{d\xi}{ds} & \frac{d\eta}{ds} \\ A & 2B & C \end{vmatrix} = C \left(\frac{d\xi}{ds}\right)^2 - 2B \left(\frac{d\xi}{ds}\right) \left(\frac{d\eta}{ds}\right) + A \left(\frac{d\eta}{ds}\right)^2 \quad (1.5)$$

is zero. If the determinant Δ is not zero, all the higher partials can be solved for by successive differentiations of known quantities with respect to s , the distance along the boundary, and the resulting Taylor's series will uniquely specify the resulting solution, within some finite area of convergence. Thus, we have shown that Cauchy conditions on a boundary do choose a particular solution unless the boundary is such that the determinant Δ is zero along it.

The equation $\Delta = 0$ is the equation of a curve

$$C(x, y)(dx)^2 - 2B(x, y)dxdy + A(x, y)(dy)^2 = 0 \quad (1.6)$$

(where we have changed the differentials $d\xi, d\eta$ into the more familiar dx, dy or, rather, of two families of curves, for this equation may be factored, giving

$$Ady = (B + \sqrt{B^2 - AC})dx ; Ady = (B - \sqrt{B^2 - AC})dx \quad (1.7)$$

These curves are characteristic of the partial differential equation (1.7) and are called the characteristics of the equation. If the boundary line happens to coincide with one of them, then specifying Cauchy conditions along it will not uniquely specify the solution; if the boundary cuts each curve of each family once, then Cauchy conditions along it will uniquely specify a solution.

1.3 Existence and Uniqueness

Before attempting to solve a problem involving a PDE we would like to know if a solution exists, and, if it exists, if the solution is unique. Also, in problems involving time, whether a solution exists $\forall t > 0$ (global existence) or only up to a given value of t , i.e., only for $0 < t < t_0$ (finite time blow-up, shock formation). As well as the equation there could be certain boundary and initial conditions. We would also like to know whether the solution of the problem depends continuously of the prescribed data, i.e., small changes in boundary or initial conditions produce only small changes in the solution.

As illustration from ordinary differential equations (ODEs) we have the following simple problems [75]:

Example-1 $\frac{du}{dt} = u, u(0) = 1.$

Solution: $u = e^t$ exists for $0 < t < \infty.$

Example-2 $\frac{du}{dt} = u^2, u(0) = 1.$

Solution: $u = \frac{1}{(1-t)}$ exists for $0 \leq t < 1$.

Example-3 $\frac{du}{dt} = \sqrt{u}$, $u(0) = 0$.

Solution: This example has two solutions: $u \equiv 0$ and $u = \frac{t^2}{4}$ (non uniqueness).

We say that the PDE with boundary or initial conditions is well-formed (or well-posed) if its solution exists (globally), is unique and depends continuously on the assigned data. If any of these three properties (existence, uniqueness and stability) is not satisfied, the problem (PDE, BCs and ICs) is said to be ill-posed. Usually, problems involving linear systems are well-formed but this may not always be the case for nonlinear systems (bifurcation of solutions, etc.).

As illustration from PDE we have the following simple problem

Example-4

A simple example of showing uniqueness in this case is provided by [75]:

$$\nabla^2 u = F \text{ in } \Omega \text{ (Poisson's equation),}$$

with $u = 0$ on $\partial\Omega$, the boundary of Ω , and F is some given function of x .

Solution

Suppose u_1 and u_2 are two different solutions satisfying the equation and the boundary conditions. Then consider $w = u_1 - u_2$; $\nabla^2 w = 0$ in Ω and $w = 0$ on $\partial\Omega$. Now the divergence theorem gives

$$\iint_{\partial\Omega} w \nabla w \cdot \mathbf{n} dS = \iiint_{\Omega} \nabla \cdot (w \nabla w) dV = \iiint_{\Omega} \{w \nabla^2 w + (\nabla w)^2\} dV$$

Where \mathbf{n} is a unit normal- outwards from Ω .

$$\iiint_{\Omega} (\nabla w)^2 dV = \iint_{\partial\Omega} w \frac{\partial w}{\partial n} dS = 0.$$

Now the integrand $(\nabla w)^2$ is non-negative in Ω and hence for the equality to hold we must have $\nabla w \equiv 0$ in Ω ; i.e., $w = \text{constant}$ in Ω . Since $w = 0$ on $\partial\Omega$ and the solution is smooth, we must have: $w \equiv 0$ in Ω ; i.e., $u_1 = u_2$. The same proof works if $\frac{\partial u}{\partial n}$ is given on $\partial\Omega$ or for mixed conditions.

Self-adjoint eigenvalue problems [3] can be treated also by similar methods like that discussed above. In addition, Green's function and its applications to eigenvalue problems and boundary-value problems for ordinary differential equations can be investigated. Following the general theory of eigenvalues and eigenfunctions, the most common special functions, including the Bessel, Legendre, and Hermite functions, can be discussed as examples of the major role of special functions in the physical and engineering sciences.

Applications to heat conduction problems and the Schrödinger equation for the linear harmonic oscillator can be also included [1].

1.4 Important Partial Differential Equations

In this section we discuss the different types of PDEs encountered in the physical sciences and engineering which are extended to situations involving more than one independent variable.

A PDE is an equation relating an unknown function (the dependent variable) of two or more variables to its partial derivatives with respect to those variables. The most commonly occurring independent variables are those describing position and time, and so we will couch our discussion and examples in notation appropriate to them.

We will focus our attention on the equations that arise most often in physical situations. We will restrict our discussion, therefore, to linear PDEs, i.e. those of first degree in the dependent variable. Furthermore, we will discuss primarily second-order equations.

The solution of first order PDEs will necessarily be involved in treating these, and some of the methods discussed can be extended without difficulty to third- and higher-order equations. We shall also see that many ideas developed for ODEs can be carried over directly into the study of PDEs.

In the next section we will concentrate on general solutions of PDEs in terms of arbitrary functions and the particular solutions that may be derived from them in the presence of boundary conditions. We also discuss the existence and uniqueness of the solutions to PDEs under given boundary conditions. Furthermore, the methods most commonly used in practice for obtaining solutions to PDEs subject to given boundary conditions will be also considered in the next sections. These methods include the separation of variables, integral transforms and Green's functions.

This division of material is rather arbitrary and has been made only to emphasise the general usefulness of the latter methods. In particular, it will be readily apparent that some of the results of the present chapter are in fact solutions in the form of separated variables but arrived at by a different approach.

Most of the important PDEs of physics are second-order and linear. In order to gain familiarity with their general form, some of the more important ones will now be briefly discussed. These equations apply to a wide variety of different physical systems.

Since, in general, the PDEs listed below describe three-dimensional situations, the independent variables are \mathbf{r} and t , where \mathbf{r} is the position vector and t is time. The actual variables used to specify the position vector \mathbf{r} are dictated by the coordinate system in use.

For example, in Cartesian coordinates the independent variables of position are x , y and z , whereas in spherical polar coordinates they are r , θ and ϕ . The equations may be written in a coordinate-independent manner, however, by the use of the Laplacian operator ∇^2 .

Among the most frequently encountered PDEs are the following [76-78]:

1.4-1 The wave equation

The wave equation

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (1.8)$$

describes the displacement from equilibrium, $u(\mathbf{r}, t)$, of a vibrating string or membrane or a vibrating solid, gas or liquid, as a function of position and time. The equation also occurs in electromagnetism, where u may be a component of the electric or magnetic field in an electromagnetic wave or the current or voltage along a transmission line. The quantity c is the speed of propagation of the waves.

1.4-2 The diffusion equation

The diffusion equation

$$k \nabla^2 u = \frac{\partial u}{\partial t} \quad (1.9)$$

describes the temperature u in a region containing no heat sources or sinks; it also applies to the diffusion of a chemical that has a concentration $u(\mathbf{r}, t)$. The constant k is called the diffusivity. The equation is clearly second order in the three spatial variables, but first order in time.

The Helmholtz and time-independent diffusion equations takes the form

$$\nabla^2 \psi \pm k^2 = 0. \quad (1.10)$$

These equations appear in such diverse phenomena as

- a. elastic waves in solids including vibrating strings, bars, membranes,
- b. sound or acoustics,
- c. electromagnetic waves, and
- d. nuclear reactors.

1.4-3 The Laplace's equation

The Laplace's equation is given by

$$\nabla^2 \psi = 0. \quad (1.11)$$

This very common and very important equation occurs in the studies of electromagnetic phenomena including electrostatics, dielectrics, steady currents, and magnetostatics, hydrodynamics (irrotational flow of perfect fluid and surface waves), heat flow, and gravitation.

1.4-4 The Poisson's equation

The Poisson's equation is given by

$$\nabla^2 u = \rho(\mathbf{r}). \quad (1.12)$$

It describes the same physical situations as Laplace's equation, but in regions containing matter, charges or sources of heat or fluid. The function $\rho(\mathbf{r})$ is called the source density and in physical applications usually contains some multiplicative physical constants. For example, if u is the electrostatic potential in some region of space, in which case ρ is the density of electric charge, then $\nabla^2 u = -\rho(\mathbf{r})/\epsilon_0$, where ϵ_0 is the permittivity of free space. Alternatively, u might represent the gravitational potential in some region where the matter density is given by ρ ; then $\nabla^2 u = 4\pi G\rho(\mathbf{r})$, where G is the gravitational constant. In contrast to the homogeneous Laplace equation, Poisson's equation is non-homogeneous with a source term.

1.4-5 The Schrödinger's equation

The Schrödinger's equation

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{r})u = i\hbar\frac{\partial u}{\partial t}, \quad (1.13)$$

describes the quantum mechanical wave function $u(\mathbf{r}, t)$ of a non-relativistic particle of mass m ; \hbar is Planck's constant divided by 2π . Like the diffusion equation it is second order in the three spatial variables and first order in time. For the time-independent case we have

$$-\frac{\hbar^2}{2m}\nabla^2 \psi + V\psi = E\psi \quad (1.14)$$

which is the stationary Schrödinger's equation.

1.4-6 The Klein-Gordon equation

The Klein-Gordon equation is given by

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0.$$

It is most often written in natural units as

$$-\partial_t^2 \psi + \nabla^2 \psi = m^2 \psi$$

The form is determined by requiring that plane wave solutions of the equation

$$\psi = e^{-i\omega t + ikx} = e^{ik_\mu x^\mu}$$

obey the energy momentum relation of special relativity:

$$-p_\mu p^\mu = E^2 - P^2 = \omega^2 - k^2 = -k_\mu k^\mu = m^2$$

Unlike the Schrödinger equation, there are two values of ω for each k , one positive and one negative. Only by separating out the positive and negative frequency parts does the equation describe a relativistic wave function. For the time-independent case, the Klein–Gordon equation becomes

$$\left[\nabla^2 - \frac{m^2 c^2}{\hbar^2} \right] \psi(\mathbf{r}) = 0$$

which is the homogeneous screened Poisson equation.

1.5 General form of solution

Before turning to the methods by which we may hope to solve PDEs such as those listed in the previous section, it is instructive, to study how PDEs may be formed from a set of possible solutions. Such a study can provide an indication of how equations obtained not from possible solutions but from physical arguments might be solved.

For definiteness let us suppose we have a set of functions involving two independent variables x and y . Without further specification this is of course a very wide set of functions, and we could not expect to find a useful equation that they all satisfy. However, let us consider a type of function $u_i(x, y)$ in which x and y appear in a particular way, such that u_i can be written as a function (however complicated) of a single variable p , itself a simple function of x and y .

Let us illustrate this by considering the three functions [79]

$$u_1(x, y) = x^4 + 4(x^2 y + y^2 + 1),$$

$$u_2(x, y) = \sin x^2 \cos 2y + \cos x^2 \sin 2y,$$

$$u_3(x, y) = \frac{x^2 + 2y + 2}{3x^2 + 6y + 5}.$$

These are all fairly complicated functions of x and y and a single differential equation of which each one is a solution is not obvious. However, if we observe that in fact each can be expressed as a function of the variable $p = x^2 + 2y$ alone (with no other x or y involved) then a great simplification takes place. Written in terms of p the above equations become

$$u_1(x, y) = (x^2 + 2y)^2 + 4 = p^2 + 4 = f_1(p),$$

$$u_2(x, y) = \sin(x^2 + 2y) = \sin p = f_2(p),$$

$$u_3(x, y) = \frac{(x^2 + 2y) + 2}{3(x^2 + 2y) + 5} = \frac{p + 2}{3p + 5} = f_3(p).$$

Let us now form, for each u_i , the partial derivatives $\partial u_i / \partial x$ and $\partial u_i / \partial y$. In each case these are (writing both the form for general p and the one appropriate to our particular case, $p = x^2 + 2y$)

$$\frac{\partial u_i}{\partial x} = \frac{df_i(p)}{dp} \frac{\partial p}{\partial x} = 2x f_i',$$

$$\frac{\partial u_i}{\partial y} = \frac{df_i(p)}{dp} \frac{\partial p}{\partial y} = 2f_i',$$

for $i = 1, 2, 3$. All reference to the form of f_i can be eliminated from these equations by cross-multiplication, obtaining

$$\frac{\partial p}{\partial y} \frac{\partial u_i}{\partial x} = \frac{\partial p}{\partial x} \frac{\partial u_i}{\partial y},$$

or, for our specific form, $p = x^2 + 2y$,

$$\frac{\partial u_i}{\partial x} = x \frac{\partial u_i}{\partial y}. \quad (1.15)$$

It is thus apparent that not only are the three functions u_1, u_2, u_3 solutions of the PDE (1.15) but so also is any arbitrary function $f(p)$ of which the argument p has the form $x^2 + 2y$.

1.6 General and Particular Solutions

In the last section we found that the first order PDE (1.15) has as a solution any function of the variable: $x^2 + 2y$. This points the way for the solution of PDEs of other orders, as follows. It is not generally true that an n th-order PDE can always be considered as resulting from the elimination of n arbitrary functions from its solution (as opposed to the elimination of n arbitrary constants for an n th-order ODE). However, given specific PDEs we can try to solve them by seeking combinations of variables in terms of which the solutions may be expressed as arbitrary functions. Where this is possible, we may expect n combinations to be involved in the solution.

Naturally, the exact functional form of the solution for any particular situation must be determined by some set of boundary conditions. For instance, if the PDE contains two independent variables x and y then for complete determination of its solution the boundary conditions will take a form equivalent to specifying $u(x, y)$ along a suitable continuum of points in the xy -plane (usually along a line).

We now discuss the general and particular solutions of first- and second order PDEs. In order to simplify the algebra, we will restrict our discussion to equations containing just two independent variables x and y . Nevertheless, the method presented below may be extended to equations containing several independent variables.

1.6-1 First-Order Equations

Although most of the PDEs encountered in physical contexts are second order (i.e. they contain $\partial^2 u / \partial x^2$ or $\partial^2 u / \partial x \partial y$, etc.), we now discuss first-order equations to illustrate the general considerations involved in the form of the solution and in satisfying any boundary conditions on the solution.

The most general first-order linear PDE (containing two independent variables) is of the form [79]

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} + C(x, y) u = R(x, y), \quad (1.16)$$

where $A(x, y)$, $B(x, y)$, $C(x, y)$ and $R(x, y)$ are given functions. Clearly, if either $A(x, y)$ or $B(x, y)$ is zero then the PDE may be solved straightforwardly as a first-order linear ODE, the only modification being that the arbitrary constant of integration becomes an arbitrary function of x or y , respectively.

When the PDE contains partial derivatives with respect to both independent variables then, of course, we cannot employ the above procedure but must seek an alternative method. Let us for the moment restrict our attention to the special case in which $C(x, y) = R(x, y) = 0$ and, following the discussion of the previous section, look for solutions of the form $u(x, y) = f(p)$, where p is some, at present unknown, combination of x and y . We then have

$$\frac{\partial u}{\partial x} = \frac{df(p)}{dp} \frac{\partial p}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{df(p)}{dp} \frac{\partial p}{\partial y},$$

which, when substituted into the PDE (1.16), give

$$\left[A(x, y) \frac{\partial p}{\partial x} + B(x, y) \frac{\partial p}{\partial y} \right] \frac{df(p)}{dp} = 0.$$

This removes all reference to the actual form of the function $f(p)$ since for non-trivial p we must have

$$A(x, y) \frac{\partial p}{\partial x} + B(x, y) \frac{\partial p}{\partial y} = 0. \quad (1.17)$$

Let us now consider the necessary condition for $f(p)$ to remain constant as x and y vary; this is that p itself remains constant. Thus for f to remain constant implies that x and y must vary in such a way that

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = 0. \quad (1.18)$$

The forms of (1.17) and (1.18) are very alike and become the same if we require that [79]

$$\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)}. \quad (1.19)$$

By integrating this expression, the form of p can be found.

Example

For example, let us find two solutions of

$$x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0, \quad (1.20)$$

(i): the first of which takes the value $2y + 1$ on the line $x = 1$, and

(ii): the second takes the value 4 at the point $(1, 1)$.

If we seek a solution of the form $u(x, y) = f(p)$, we deduce from (1.19) that $u(x, y)$ will be constant along lines of (x, y) that satisfy

$$\frac{dx}{x} = \frac{dy}{-2y},$$

which on integrating gives $x = cy^{-\frac{1}{2}}$. Identifying the constant of integration c with $p^{\frac{1}{2}}$ (to avoid fractional powers), we conclude that $p = x^2y$. Thus, the general solution of the PDE (1.20) is

$$u(x, y) = f(x^2y),$$

where f is an arbitrary function. We must now find the particular solutions that obey each of the imposed boundary conditions. For boundary condition (i) a little thought shows that the particular solution required is

$$u(x, y) = 2(x^2y) + 1 = 2x^2y + 1. \quad (1.21)$$

For boundary condition (ii) some obviously acceptable solutions are

$$u(x, y) = x^2y + 3,$$

$$u(x, y) = 4x^2y,$$

$$u(x, y) = 4.$$

Each is a valid solution (the freedom of choice of form arises from the fact that u is specified at only one point $(1, 1)$, and not along a continuum (say), as in boundary condition (i)). All three are particular examples of the general solution, which may be written, for example, as

$$u(x, y) = x^2y + 3 + g(x^2y),$$

where $g = g(x^2y) = g(p)$ is an arbitrary function subject only to $g(1) = 0$. For this example, the forms of g corresponding to the particular solutions listed above are

$$g(p) = 0, g(p) = 3p - 3, g(p) = 1 - p.$$

As mentioned above, in order to find a solution of the form $u(x, y) = f(p)$ we require that the original PDE contains no term in u , but only terms containing its partial derivatives. If a term in u is present, so that $C(x, y) \neq 0$ in (1.16), then the procedure needs some modification, since we cannot simply divide out the dependence on $f(p)$ to obtain (1.17). In such cases we look instead for a solution of the form $u(x, y) = h(x, y)f(p)$. We illustrate this method in the following example [79]:

Example: Find the general solution of

$$x \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} - 2u = 0. \quad (1.22)$$

We seek a solution of the form $u(x, y) = h(x, y)f(p)$, with the consequence that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial h}{\partial x} f(p) + h \frac{df(p)}{dp} \frac{\partial p}{\partial x}, \\ \frac{\partial u}{\partial y} &= \frac{\partial h}{\partial y} f(p) + h \frac{df(p)}{dp} \frac{\partial p}{\partial y}. \end{aligned}$$

Substituting these expressions into the PDE (1.22) and rearranging, we obtain

$$\left(x \frac{\partial h}{\partial x} + 2 \frac{\partial h}{\partial y} - 2h \right) f(p) + \left(x \frac{\partial p}{\partial x} + 2 \frac{\partial p}{\partial y} \right) h \frac{df(p)}{dp} = 0,$$

The first factor in parentheses is just the original PDE with u replaced by h . Therefore, if h is any solution of the PDE, however simple, this term will vanish, to leave

$$\left(x \frac{\partial p}{\partial x} + 2 \frac{\partial p}{\partial y} \right) h \frac{df(p)}{dp} = 0,$$

from which, as in the previous case, we obtain

$$x \frac{\partial p}{\partial x} + 2 \frac{\partial p}{\partial y} = 0.$$

From (1.18) and (1.19) we see that $u(x, y)$ will be constant along lines of (x, y) that satisfy

$$\frac{dx}{x} = \frac{dy}{2},$$

which integrates to give $x = c \exp(y/2)$. Identifying the constant of integration c with p we find $p = x \exp(-y/2)$. Thus, the general solution of (1.22) is

$$u(x, y) = h(x, y) f \left(x \exp \left(-\frac{1}{2} y \right) \right),$$

where $f(p)$ is any arbitrary function of p and $h(x, y)$ is any solution of (1.22).

If we take, for example, $h(x, y) = \exp(y)$, which clearly satisfies (1.22), then the general solution is

$$u(x, y) = \{ \exp(y) \} f \left(x \exp \left(-\frac{1}{2} y \right) \right).$$

Alternatively, $h(x, y) = x^2$ also satisfies (1.22) and so the general solution to the equation can also be written

$$u(x, y) = x^2 g\left(x \exp\left(-\frac{1}{2} y\right)\right),$$

where g is an arbitrary function of p ; clearly $g(p) = \frac{f(p)}{p^2}$.

1.6-2 Second-Order Equations

As noted in section 1.4, second-order linear PDEs are of great importance in describing the behavior of many physical systems. As in our discussion of first order equations, for the moment we shall restrict our discussion to equations with just two independent variables; extensions to a greater number of independent variables are straightforward.

The most general second-order linear PDE (containing two independent variables) has the form [79]

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = R(x, y), \quad (1.23)$$

where A, B, \dots, F and $R(x, y)$ are given functions of x and y . Because of the nature of the solutions to such equations, they are usually divided into three classes, a division of which we will make further use in subsection 1.7-2.

The equation (1.23) is called hyperbolic if $B^2 > 4AC$, parabolic if $B^2 = 4AC$ and elliptic if $B^2 < 4AC$. Clearly, if A, B and C are functions of x and y (rather than just constants) then the equation might be of different types in different parts of the xy -plane.

Equation (1.23) obviously represents a very large class of PDEs, and it is usually impossible to find closed-form solutions to most of these equations. Therefore, for the moment we shall consider only homogeneous equations, with $R(x, y) = 0$, and make the further (greatly simplifying) restriction that, throughout the remainder of this section, A, B, \dots, F are not functions of x and y but merely constants.

We now tackle the problem of solving some types of second order PDEs with constant coefficients by seeking solutions that are arbitrary functions of particular combinations of independent variables [79], just as we did for first-order equations.

Following the discussion of the previous section, we can hope to find such solutions only if all the terms of the equation involve the same total number of differentiations, i.e. all terms are of the same order, although the number of differentiations with respect to the individual independent variables may be different. This means that in (1.23) we require the constants D, E and F to be identically zero (we have, of course, already assumed that $R(x, y)$ is zero), so that we are now considering only equations of the form

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0, \quad (1.24)$$

where A, B and C are constants. We note that both the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0,$$

and the two-dimensional Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

are of this form, but that the diffusion equation,

$$k \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0,$$

is not, since it contains a first-order derivative.

Since all the terms in (1.24) involve two differentiations, by assuming a solution of the form $u(x, y) = f(p)$, where p is some unknown function of x and y (or t), we may be able to obtain a common factor $d^2 f(p)/dp^2$ as the only appearance of f on the LHS. Then, because of the zero RHS, all reference to the form of f can be cancelled out.

We can gain some guidance on suitable forms for the combination $p = p(x, y)$ by considering $\partial u/\partial x$ when u is given by $u(x, y) = f(p)$, for then

$$\frac{\partial u}{\partial x} = \frac{df(p)}{dp} \frac{\partial p}{\partial x}.$$

Clearly differentiation of this equation with respect to x (or y) will not lead to a single term on the RHS, containing f only as $d^2 f(p)/dp^2$, unless the factor $\partial p/\partial x$ is a constant so that $\partial^2 p/\partial x^2$ and $\partial^2 p/\partial x \partial y$ are necessarily zero. This shows that p must be a linear function of x . In an exactly similar way p must also be a linear function of y , i.e. $p = ax + by$.

If we assume a solution of (1.24) of the form [79]: $u(x, y) = f(ax + by)$, and evaluate the terms ready for substitution into (1.24), we obtain

$$\begin{aligned} \frac{\partial u}{\partial x} &= a \frac{df(p)}{dp}, & \frac{\partial u}{\partial y} &= b \frac{df(p)}{dp}, \\ \frac{\partial^2 u}{\partial x^2} &= a^2 \frac{d^2 f(p)}{dp^2}, & \frac{\partial^2 u}{\partial x \partial y} &= ab \frac{d^2 f(p)}{dp^2}, & \frac{\partial^2 u}{\partial y^2} &= b^2 \frac{d^2 f(p)}{dp^2}, \end{aligned}$$

which on substitution give

$$(Aa^2 + Bab + Cb^2) \frac{d^2 f(p)}{dp^2} = 0. \quad (1.25)$$

This is the form we have been seeking, since now a solution independent of the form of f can be obtained if we require that a and b satisfy

$$Aa^2 + Bab + Cb^2 = 0.$$

From this quadratic equation, two values for the ratio of the two constants a and b are obtained,

$$b/a = [-B \pm (B^2 - 4AC)^{1/2}]/2C.$$

If we denote these two ratios by λ_1 and λ_2 then any functions of the two variables

$$p_1 = x + \lambda_1 y, \quad p_2 = x + \lambda_2 y$$

will be solutions of the original equation (1.24). The omission of the constant factor a from p_1 and p_2 is of no consequence since this can always be absorbed into the particular form of any chosen function; only the relative weighting of x and y in p is important.

Since p_1 and p_2 are in general different, we can thus write the general solution of (1.24) as

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y), \quad (1.26)$$

where f and g are arbitrary functions.

Finally, we note that the alternative solution $d^2 f(p)/dp^2 = 0$ to (1.25) leads only to the trivial solution $u(x, y) = kx + ly + m$, for which all second derivatives are individually zero.

1.7 Characteristics and the Existence of Solutions

So far in this chapter we have discussed how to find general solutions to various types of first- and second-order linear PDE. Moreover, given a set of boundary conditions we have shown how to find the particular solution (or class of solutions) that satisfies them. For first-order equations, for example, we found that if the value of $u(x, y)$ is specified along some curve in the xy -plane then the solution to the PDE is in general unique, but that if $u(x, y)$ is specified at only a single point then the solution is not unique: there exists a class of particular solutions all of which satisfy the boundary condition. In this section we make more rigorous the notion of the respective types of boundary condition that cause a PDE to have a unique solution, a class of solutions, or no solution at all.

1.7-1 First-Order Equations

Let us consider the general first-order PDE (1.16) but now write it as

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = F(x, y, u). \quad (1.27)$$

Suppose we wish to solve this PDE subject to the boundary condition that $u(x, y) = \phi(s)$ is specified along some curve C in the xy -plane that is described parametrically by the equations $x = x(s)$ and $y = y(s)$, where s is the arc length along C . The variation of u along C is therefore given by

$$\frac{du}{ds} = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = \frac{d\phi}{ds}. \quad (1.28)$$

We may then solve the two (inhomogeneous) simultaneous linear equations (1.27) and (1.28) for $\partial u/\partial x$ and $\partial u/\partial y$, unless the determinant of the coefficients vanishes, i.e. unless

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ A & B \end{vmatrix} = 0.$$

At each point in the xy -plane this equation determines a set of curves called characteristic curves (or just characteristics), which thus satisfy

$$B \frac{dx}{ds} - A \frac{dy}{ds} = 0,$$

or, multiplying through by ds/dx and dividing through by A ,

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)}. \quad (1.29)$$

However, we have already met (1.22) in subsection 1.6-1 on first order PDEs, where solutions of the form $u(x,y) = f(p)$, where p is some combination of x and y , were discussed. Comparing (1.29) with (1.19) we see that the characteristics are merely those curves along which p is constant.

Since the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ may be evaluated provided the boundary curve C does not lie along a characteristic, defining $u(x,y) = \phi(s)$ along C is sufficient to specify the solution to the original problem (equation plus boundary conditions) near the curve C , in terms of a Taylor expansion about C . Therefore, the characteristics can be considered as the curves along which information about the solution $u(x,y)$ ‘propagates’.

1.7-2 Second-Order Equations

The concept of characteristics can be extended naturally to second- (and higher-) order equations. In this case let us write the general second-order linear PDE (1.23) as

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2} = F\left(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \quad (1.30)$$

For second-order equations we might expect that relevant boundary conditions would involve specifying u , or some of its first derivatives, or both, along a suitable set of boundaries bordering or enclosing the region over which a solution is sought. Three common types of boundary condition occur and are associated with the names of Dirichlet, Neumann and Cauchy [79]. They are as follows.

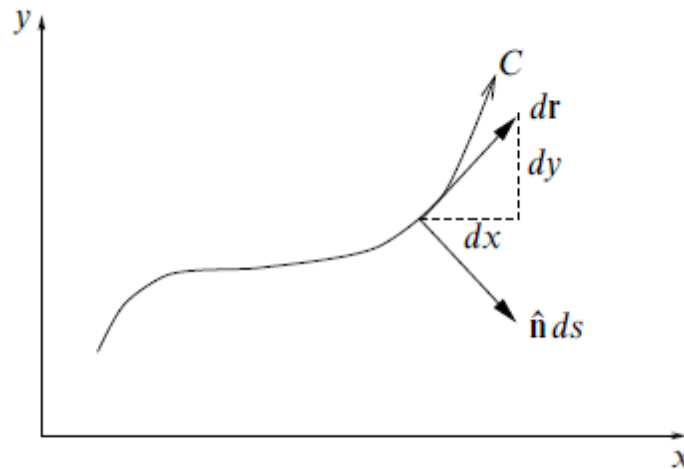


Figure-1.2 A boundary curve C and its tangent and unit normal at a given point.

Dirichlet: The value of u is specified at each point of the boundary.

Neumann: The value of $\partial u/\partial n$, the normal derivative of u , is specified at each point of the boundary. Note that $\partial u/\partial n = \nabla u \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal to the boundary at each point.

Cauchy: Both u and $\partial u/\partial n$ are specified at each point of the boundary.

Let us consider for the moment the solution of (1.30) subject to the Cauchy boundary conditions, i.e. u and $\partial u/\partial n$ are specified along some boundary curve C in the xy -plane defined by the parametric equations $x = x(s)$, $y = y(s)$, s being the arc length along C (see Figure-1.2). Let us suppose that along C , we have $u(x, y) = \phi(s)$ and $\partial u/\partial n = \psi(s)$. At any point on C the vector $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j}$ is a tangent to the curve and $\hat{\mathbf{n}} ds = dy \mathbf{i} - dx \mathbf{j}$ is a vector normal to the curve. Thus on C we have

$$\frac{\partial u}{\partial s} \equiv \nabla u \cdot \frac{d\mathbf{r}}{ds} = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = \frac{d\phi(s)}{ds},$$

$$\frac{\partial u}{\partial n} \equiv \nabla u \cdot \hat{\mathbf{n}} = \frac{\partial u}{\partial x} \frac{dy}{ds} - \frac{\partial u}{\partial y} \frac{dx}{ds} = \psi(s).$$

These two equations may then be solved straightforwardly for the first partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ along C . Using the chain rule to write

$$\frac{d}{ds} = \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y},$$

we may differentiate the two first derivatives $\partial u/\partial x$ and $\partial u/\partial y$ along the boundary to obtain the pair of equations

$$\frac{d}{ds} \left(\frac{\partial u}{\partial x} \right) = \frac{dx}{ds} \frac{\partial^2 u}{\partial x^2} + \frac{dy}{ds} \frac{\partial^2 u}{\partial x \partial y},$$

$$\frac{d}{ds} \left(\frac{\partial u}{\partial y} \right) = \frac{dx}{ds} \frac{\partial^2 u}{\partial x \partial y} + \frac{dy}{ds} \frac{\partial^2 u}{\partial y^2}.$$

We may now solve these two equations, together with the original PDE (1.30), for the second partial derivatives of u , except where the determinant of their coefficients equals zero,

$$\begin{vmatrix} A & B & C \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \end{vmatrix} = 0.$$

Expanding out the determinant,

$$A \left(\frac{dy}{ds} \right)^2 - B \left(\frac{dx}{ds} \right) \left(\frac{dy}{ds} \right) + C \left(\frac{dx}{ds} \right)^2 = 0.$$

Multiplying through by $\left(\frac{ds}{dx} \right)^2$ we obtain

$$A \left(\frac{dy}{dx} \right)^2 - B \frac{dy}{dx} + C = 0. \quad (1.31)$$

which is the ODE for the curves in the xy -plane along which the second partial derivatives of u cannot be found.

As for the first-order case, the curves satisfying (1.31) are called characteristics of the original PDE. These characteristics have tangents at each point given by (when $A \neq 0$)

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \quad (1.32)$$

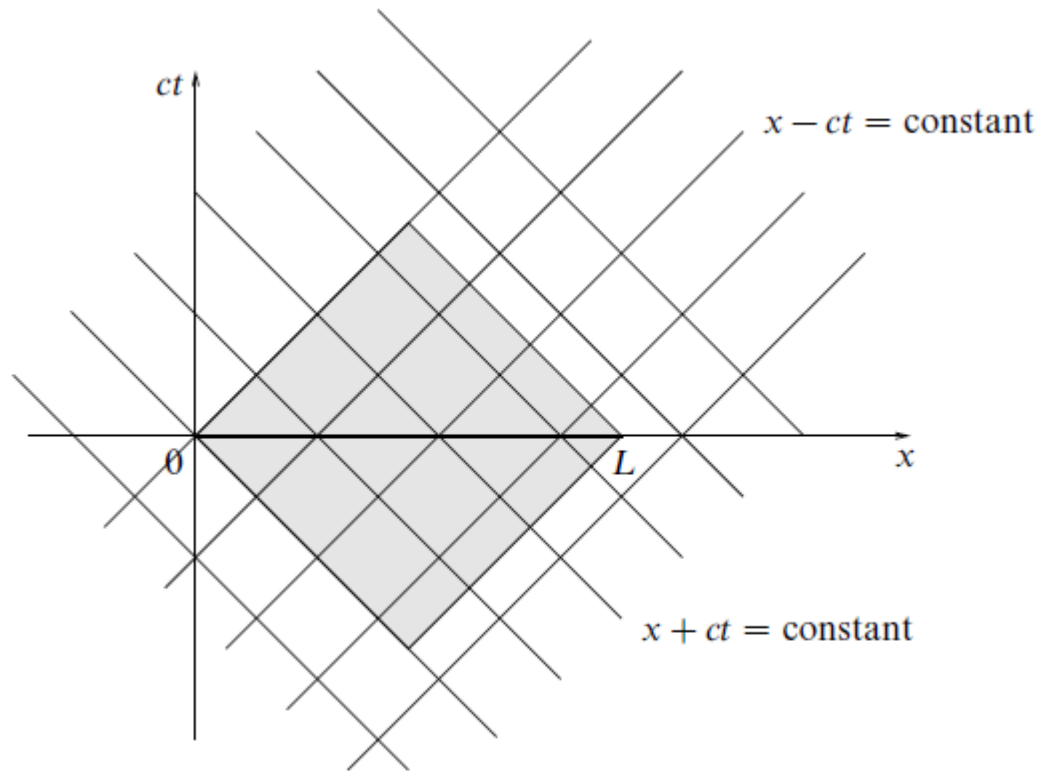


Figure-1.3 The characteristics for the one-dimensional wave equation. The shaded region indicates the region over which the solution is determined by specifying Cauchy boundary conditions at $t = 0$ on the line segment $x = 0$ to $x = L$.

Clearly, when the original PDE is hyperbolic ($B^2 > 4AC$), equation (1.3) defines two families of real curves in the xy -plane; when the equation is parabolic ($B^2 = 4AC$) it defines one family of real curves; and when the equation is elliptic ($B^2 < 4AC$) it defines two families of complex curves. Furthermore, when A , B and C are constants, rather than functions of x and y , the equations of the characteristics will be of the form $x + \lambda y = \text{constant}$.

As an example, let us find the characteristics of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0.$$

This is a hyperbolic equation with $A = 1$, $B = 0$ and $C = -\frac{1}{c^2}$. Therefore, from (1.31) the characteristics are given by

$$\left(\frac{dx}{dt}\right)^2 = c^2,$$

and so the characteristics are the straight lines $x - ct = \text{constant}$ and $x + ct = \text{constant}$, see Figure-1.3.

The characteristics of second order PDEs can be considered as the curves along which partial information about the solution $u(x, y)$ ‘propagates’. Consider a point in the space that has the independent variables as its coordinates; unless both of the two characteristics that pass through the point intersect the curve along which the boundary conditions are specified, the solution will not be determined at that point. In particular, if the equation is hyperbolic, so that we obtain two families of real characteristics in the xy -plane, then Cauchy boundary conditions propagate partial information concerning the solution along the characteristics, belonging to each family, that intersect the boundary curve C . The solution u is then specified in the region common to these two families of characteristics. For instance, the characteristics of the hyperbolic one-dimensional wave equation in the last example are shown in Figure-1.3. By specifying Cauchy boundary conditions u and $\partial u/\partial t$ on the line segment $t = 0$, $x = 0$ to L , the solution is specified in the shaded region.

As in the case of first order PDEs, however, problems can arise. For example, if for a hyperbolic equation the boundary curve intersects any characteristic more than once then Cauchy conditions along C can overdetermine the problem, resulting in there being no solution. In this case either the boundary curve C must be altered, or the boundary conditions on the offending parts of C must be relaxed to Dirichlet or Neumann conditions.

Table 1.1 The appropriate boundary conditions for different types of partial differential equation

Equation type	Boundary	Boundary Conditions
hyperbolic	open	Cauchy
parabolic	open	Dirichlet or Neumann
elliptic	closed	Dirichlet or Neumann

The general considerations involved in deciding which boundary conditions are appropriate for a particular problem are complex, and we do not discuss them any further here. We merely note that whether the various types of boundary condition are appropriate (in that they give a solution that is unique, sometimes to within a constant, and is well defined) depends upon the type of second-order equation under consideration and on whether the region of solution is bounded by a closed or an open curve (or a surface if there are more than two independent variables) [79]. Note that part of a closed boundary may be at infinity if conditions are imposed on u or $\partial u/\partial n$ there.

It may be shown that the appropriate boundary-condition and equation-type pairings are as given in Table 1.1.

For example, Laplace's equation $\nabla^2 u = 0$ is elliptic and thus requires either Dirichlet or Neumann boundary conditions on a closed boundary which, as we have already noted, may be at infinity if the behavior of u is specified there (most often u or $\partial u/\partial n \rightarrow 0$ at infinity).

1.8 Separation of Variables

In the previous sections we demonstrated the methods by which general solutions of some PDEs may be obtained in terms of arbitrary functions. In particular, solutions containing the independent variables in definite combinations were sought, thus reducing the effective number of them. Furthermore, the equations of mathematical physics listed in the above sections are all PDEs. Our first technique for their solution splits the PDE of n variables into n ODEs. Each separation introduces an arbitrary constant of separation. If we have n variables, we have to introduce $n - 1$ constants, determined by the conditions imposed in the problem being solved [77].

In the present section we begin by taking the opposite approach, namely that of trying to keep the independent variables as separate as possible, using the method of separation of variables. In the next section we then consider integral transform methods by which one of the independent variables may be eliminated, at least from differential coefficients. Finally, in the last section we discuss the use of Green's functions in solving inhomogeneous problems.

1.8-1 The General Method

Suppose we seek a solution $u(x, y, z, t)$ to some PDE (expressed in Cartesian coordinates). Let us attempt to obtain one that has the product form

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t). \quad (1.33)$$

A solution that has this form is said to be separable in x, y, z and t , and seeking solutions of this form is called the method of separation of variables.

As simple examples we may observe that, of the functions

$$(i) \ xyz^2 \sin bt, \quad (ii) \ xy + zt, \quad (iii) \ (x^2 + y^2)z \cos \omega t,$$

(i) is completely separable, (ii) is inseparable in that no single variable can be separated out from it and written as a multiplicative factor, whilst (iii) is separable in z and t but not in x and y .

When seeking PDE solutions of the form (1.33), we are requiring not that there is no connection at all between the functions X, Y, Z and T (for example, certain parameters may appear in two or more of them), but only that X does not depend upon y, z, t , that Y does not depend on x, z, t , and so on.

For a general PDE it is likely that a separable solution is impossible, but certainly some common and important equations do have useful solutions of this form, and we will illustrate the method of solution by studying the three-dimensional wave equation

$$\nabla^2 u(\mathbf{r}) = \frac{1}{c^2} \frac{\partial^2 u(\mathbf{r})}{\partial t^2}. \quad (1.34)$$

We will work in Cartesian coordinates for the present and assume a solution of the form (1.33). Expressed in Cartesian coordinates (1.34) takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}; \quad (1.35)$$

substituting (1.33) gives

$$\frac{d^2 X}{dx^2} YZT + X \frac{d^2 Y}{dy^2} ZT + XY \frac{d^2 Z}{dz^2} T = \frac{1}{c^2} XYZ \frac{d^2 T}{dt^2},$$

which can also be written as

$$X''YZT + XY''ZT + XYZ''T = \frac{1}{c^2} XYZT'', \quad (1.36)$$

where in each case the primes refer to the ordinary derivative with respect to the independent variable upon which the function depends. This emphasizes the fact that each of the functions X, Y, Z and T has only one independent variable and thus its only derivative is its total derivative. For the same reason, in each term in (1.36) three of the four functions are unaltered by the partial differentiation and behave exactly as constant multipliers.

If we now divide (1.36) throughout by $u = XYZT$ we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2} \frac{T''}{T}, \quad (1.37)$$

This form shows the particular characteristic that is the basis of the method of separation of variables, namely that of the four terms the first is a function of x only, the second of y only, the third of z only and the RHS a function of t only and yet there is an equation connecting them. This can only be so for all x, y, z and t if each of the terms does not in fact, despite appearances, depend upon the corresponding independent variable but is equal to a constant, the four constants being such that (1.37) is satisfied.

Since there is only one equation to be satisfied and four constants involved, there is considerable freedom in the values they may take. For the purposes of our illustrative example let us make the choice of $-l^2, -m^2, -n^2$, for the first three constants. The constant associated with $c^{-2}T''/T$ must then have the value $-\mu^2 = -(l^2 + m^2 + n^2)$.

Having recognized that each term of (1.37) is individually equal to a constant (or parameter), we can now replace (1.37) by four separate ODEs:

$$\frac{X''}{X} = -l^2, \quad \frac{Y''}{Y} = -m^2, \quad \frac{Z''}{Z} = -n^2, \quad \frac{1}{c^2} \frac{T''}{T} = -\mu^2. \quad (1.38)$$

The important point to notice is not the simplicity of the equations (1.38) (the corresponding ones for a general PDE are usually far from simple) but that, by the device of assuming a separable solution, a partial differential equation (1.35), containing derivatives with respect to the four independent variables all in one equation, has been reduced to four separate ODEs (1.38). The ordinary equations are connected through four constant parameters that satisfy an algebraic relation. These constants are called separation constants. The general solutions of the equations (1.38) can be deduced straightforwardly and are

$$\begin{aligned} X(x) &= A \exp(ilx) + B \exp(-ilx), \\ Y(y) &= C \exp(imy) + D \exp(-imy), \\ Z(z) &= E \exp(inz) + F \exp(-inz), \\ T(t) &= G \exp(ic\mu t) + H \exp(-ic\mu t), \end{aligned} \quad (1.39)$$

where A, B, \dots, H are constants, which may be determined if boundary conditions are imposed on the solution. Depending on the geometry of the problem and any boundary conditions, it is sometimes more appropriate to write the solutions (1.39) in the alternative form

$$\begin{aligned} X(x) &= A' \cos lx + B' \sin lx, \\ Y(y) &= C' \cos my + D' \sin my, \\ Z(z) &= E' \cos nz + F' \sin nz, \\ T(t) &= G' \cos(c\mu t) + H' \sin(c\mu t), \end{aligned} \quad (1.40)$$

for some different set of constants A', B', \dots, H' . Clearly the choice of how best to represent the solution depends on the problem being considered. As an example, suppose that we take as particular solutions the four functions

$$X(x) = \exp(ilx), \quad Y(y) = \exp(imy), \quad Z(z) = \exp(inz), \quad T(t) = \exp(-ic\mu t).$$

This gives a particular solution of the original PDE (1.35)

$$\begin{aligned} u(x, y, z, t) &= \exp(ilx) \exp(imy) \exp(inz) \exp(-ic\mu t) \\ &= \exp[i(lx + my + nz - c\mu t)], \end{aligned}$$

which represents a plane wave of unit amplitude propagating in a direction given by the vector with components l, m, n in a Cartesian coordinate system. In the conventional notation of wave theory, l, m and n are the components of the wave-number vector \mathbf{k} , whose magnitude is given by $k = 2\pi/\lambda$, where λ is the wavelength of the wave; $c\mu$ is the angular frequency ω of the wave. This gives the equation in the form

$$u(x, y, z, t) = \exp[i(k_x x + k_y y + k_z z - \omega t)] = \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)],$$

and makes the exponent dimensionless.

The method of separation of variables can be applied to many commonly occurring PDEs encountered in physical applications.

For the one-dimensional diffusion equation

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad (1.41)$$

a solution that tends to zero as $t \rightarrow \infty$ for all x , can be easily given by

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x) \exp(-\lambda^2 \kappa t), \quad (1.42)$$

where λ is an arbitrary constant to be determined from the boundary conditions. In order to satisfy the boundary condition $u \rightarrow 0$ as $t \rightarrow \infty$, $\lambda^2 \kappa$ must be > 0 . Since κ is real and > 0 , this implies that λ is a real non-zero number and that the solution is sinusoidal in x and is not disguised hyperbolic function; this was our reason for choosing the separation constant as $-\lambda^2$.

For the two-dimensional Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad (1.43)$$

the general solution becomes

$$u(x, y) = (A \cosh \lambda x + B \sinh \lambda x)(C \cos \lambda y + D \sin \lambda y) \quad (1.44)$$

where $\lambda^2 > 0$

An alternative form, in which the exponentials are written explicitly, may be useful for other geometries or boundary conditions:

$$u(x, y) = [A \exp \lambda x + B \exp(-\lambda x)](C \cos \lambda y + D \sin \lambda y), \quad (1.45)$$

with different constants A and B .

If $\lambda^2 < 0$ then the roles of x and y interchange. The particular combination of sinusoidal and hyperbolic functions and the values of λ allowed will be determined by the geometrical properties of any specific problem, together with any prescribed or necessary boundary conditions.

1.8-2 Superposition of Separated Solutions

It will be noticed in the previous two examples in the last section that there is considerable freedom in the values of the separation constant λ , the only essential requirement being that λ has the same value in both parts of the solution, i.e. the part depending on x and the part depending on y (or t). This is a general feature for solutions in separated form, which, if the original PDE has n independent variables, will contain $n - 1$ separation constants. All that is required in general is that we associate the correct function of one independent variable with the appropriate functions of the others, the correct function being the one with the same values of the separation constants.

If the original PDE is linear (as are the Laplace, Schrodinger, diffusion and wave equations) then mathematically acceptable solutions can be formed by superposing solutions corresponding to different allowed values of the separation constants. To take a two-variable example: if

$$u_{\lambda_1}(x, y) = X_{\lambda_1}(x)Y_{\lambda_1}(y),$$

is a solution of a linear PDE obtained by giving the separation constant the value λ_1 , then the superposition

$$u(x, y) = a_1X_{\lambda_1}(x)Y_{\lambda_1}(y) + a_2X_{\lambda_2}(x)Y_{\lambda_2}(y) + \dots = \sum_i a_iX_{\lambda_i}(x)Y_{\lambda_i}(y), \quad (1.46)$$

is also a solution for any constants a_i , provided that the λ_i are the allowed values of the separation constant λ given the imposed boundary conditions. Note that if the boundary conditions allow any of the separation constants to be zero then the form of the general solution is normally different and must be deduced by returning to the separated ODEs.

The value of the superposition approach is that a boundary condition, say that $u(x, y)$ takes a particular form $f(x)$ when $y = 0$, might be met by choosing the constants a_i such that

$$f(x) = \sum_i a_iX_{\lambda_i}(x)Y_{\lambda_i}(0).$$

In general, this will be possible provided that the functions $X_{\lambda_i}(x)$ form a complete set-as do the sinusoidal functions of Fourier series or the spherical harmonics.

To explain this procedure, let us consider the following example [79]. A semi-infinite rectangular metal plate occupies the region $0 \leq x \leq \infty$ and $0 \leq y \leq b$ in the xy -plane. The temperature at the far end of the plate and along its two long sides is fixed at 0°C . If the temperature of the plate at $x = 0$ is also fixed and is given by $f(y)$, find the steady-state temperature distribution $u(x, y)$ of the plate. Hence find the temperature distribution if $f(y) = u_0$ where u_0 is a constant.

The physical situation is illustrated in Figure-1.4. With the notation we have used several times before, the two-dimensional heat diffusion equation satisfied by the temperature $u(x, y, t)$ is

$$\kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}.$$

where $\kappa = k/(C\rho)$, k being the heat conductivity, ρ the density, and C the specific heat of the medium. In this case, however, we are asked to find the steady-state temperature, which corresponds to $\partial u/\partial t = 0$, and so we are led to consider the (two-dimensional) Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We saw that assuming a separable solution of the form $u(x, y) = X(x)Y(y)$ led to solutions such as (1.44) or (1.45), or equivalent forms with x and y interchanged. In the current problem we have to satisfy the boundary conditions $u(x, 0) = 0 = u(x, b)$ and so a solution that is sinusoidal in y seems appropriate. Furthermore, since we require $u(\infty, y) = 0$ it is best to write the x -dependence of the solution explicitly in terms of exponentials rather than of hyperbolic functions. We therefore write the separable solution in the form (1.45) as

$$u(x, y) = [A \exp \lambda x + B \exp(-\lambda x)](C \cos \lambda y + D \sin \lambda y).$$

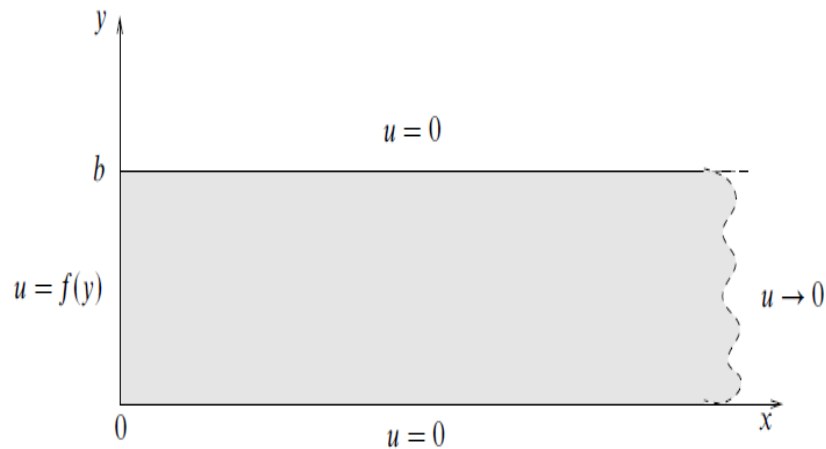


Figure-1.4 A semi-infinite metal plate whose edges are kept at fixed temperatures.

Applying the boundary conditions, we see firstly that $u(\infty, y) = 0$ implies $A = 0$ if we take $\lambda > 0$. Secondly, since $u(x, 0) = 0$ we may set $C = 0$, which, if we absorb the constant D into B , leaves us with

$$u(x, y) = B \exp(-\lambda x) \sin \lambda y.$$

But, using the condition $u(x, b) = 0$, we require $\sin \lambda b = 0$ and so λ must be equal to $n\pi/b$, where n is any positive integer.

Using the principle of superposition (1.46), the general solution satisfying the given boundary conditions can therefore be written

$$u(x, y) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right), \quad (1.47)$$

for some constants B_n . Notice that in the sum in (1.47) we have omitted negative values of n since they would lead to exponential terms that diverge as $x \rightarrow \infty$. The $n = 0$ term is also omitted since it is identically zero. Using the remaining boundary condition $u(0, y) = f(y)$ we see that the constants B_n must satisfy

$$f(y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{b}\right). \quad (1.48)$$

This is clearly a Fourier sine series expansion of $f(y)$. For (1.48) to hold, however, the continuation of $f(y)$ outside the region $0 \leq y \leq b$ must be an odd periodic function with period $2b$ (see figure 1.5). We also see from figure 1.5 that if the original function $f(y)$ does not equal zero at either of $y = 0$ and $y = b$ then its continuation has a discontinuity at the corresponding point(s); nevertheless, the Fourier series will converge to the mid-points of these jumps and hence tend to zero in this case. If, however, the top and bottom edges of the plate were held not at 0°C but at some other non-zero temperature, then, in general, the final solution would possess discontinuities at the corners $x = 0, y = 0$ and $x = 0, y = b$.

Bearing in mind these technicalities, the coefficients B_n in (1.48) are given by

$$B_n = \frac{2}{n} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy. \quad (1.49)$$

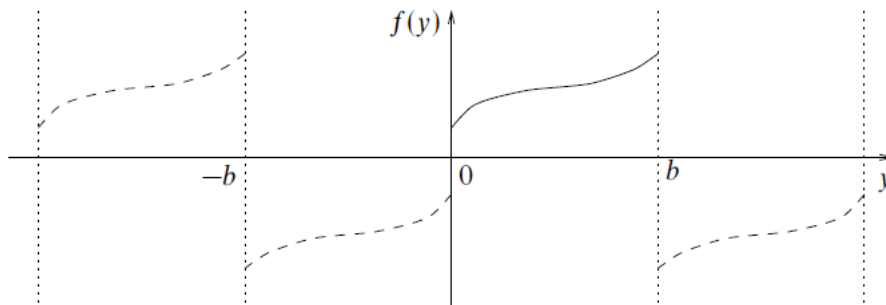


Figure-1.5 The continuation of $f(y)$ for a Fourier sine series.

Therefore, if $f(y) = u_0$ (i.e. the temperature of the side at $x = 0$ is constant along its length), (1.49) becomes

$$\begin{aligned} B_n &= \frac{2}{b} \int_0^b u_0 \sin\left(\frac{n\pi y}{b}\right) dy = \left[-\frac{2u_0}{b} \frac{b}{n\pi} \cos\left(\frac{n\pi y}{b}\right) \right]_0^b \\ &= -\frac{2u_0}{n\pi} [(-1)^n - 1] = \begin{cases} \frac{4u_0}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \end{aligned}$$

Therefore, the required solution is

$$u(x, y) = \sum_{n \text{ odd}} \frac{4u_0}{n\pi} \exp\left(-\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right).$$

In the above example the boundary conditions meant that one term in each part of the separable solution could be immediately discarded, making the problem much easier to solve. Sometimes, however, a little ingenuity is required in writing the separable solution in such a way that certain parts can be neglected immediately.

Suppose that the semi-infinite rectangular metal plate in the previous example is replaced by one that in the x -direction has finite length a . The temperature of the right-hand edge is fixed at 0°C and all other boundary conditions remain as before. Find the steady-state temperature in the plate.

As in the previous example, the boundary conditions $u(x, 0) = 0 = u(x, b)$ suggest a solution that is sinusoidal in y . In this case, however, we require $u = 0$ on $x = a$ (rather than at infinity) and so a solution in which the x -dependence is written in terms of hyperbolic functions, such as (1.44), rather than exponentials is more appropriate. Moreover, since the constants in front of the hyperbolic functions are, at this stage, arbitrary, we may write the separable solution in the most convenient way that ensures that the condition $u(a, y) = 0$ is straightforwardly satisfied. We therefore write

$$u(x, y) = [A \cosh \lambda(a - x) + B \sinh \lambda(a - x)](C \cos \lambda y + D \sin \lambda y).$$

Now the condition $u(a, y) = 0$ is easily satisfied by setting $A = 0$. As before the conditions $u(x, 0) = 0 = u(x, b)$ imply $C = 0$ and $\lambda = n\pi/b$ for integer n . Superposing the solutions for different n we then obtain

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh \left[\frac{n\pi(a-x)}{b} \right] \sin \left(\frac{n\pi y}{b} \right), \quad (1.50)$$

for some constants B_n . We have omitted negative values of n in the sum (1.50) since the relevant terms are already included in those obtained for positive n . Again the $n = 0$ term is identically zero. Using the final boundary condition $u(0, y) = f(y)$ as above we find that the constants B_n must satisfy

$$f(y) = \sum_{n=1}^{\infty} B_n \sinh \left[\frac{n\pi a}{b} \right] \sin \left(\frac{n\pi y}{b} \right),$$

and the constants B_n are given, as discussed before, by

$$B_n = \frac{2}{b \sinh(n\pi a/b)} \int_0^b f(y) \sin \left(\frac{n\pi y}{b} \right) dy. \quad (1.51)$$

For the case where $f(y) = u_0$, following the working of the previous example gives (1.51) as

$$B_n = \frac{4u_0}{n\pi \sinh(n\pi a/b)} \text{ for } n \text{ odd, } B_n = 0 \text{ for } n \text{ even} \quad (1.52)$$

The required solution is thus

$$u(x, y) = \sum_{n \text{ odd}}^{\infty} \frac{4u_0}{n\pi \sinh(n\pi a/b)} \sinh \left[\frac{n\pi(a-x)}{b} \right] \sin \left(\frac{n\pi y}{b} \right).$$

We note that, as required, in the limit $a \rightarrow \infty$ this solution tends to the solution of the previous example.

Often the principle of superposition can be used to write the solution to problems with more complicated boundary conditions as the sum of solutions to problems that each satisfy only some part of the boundary condition but when added together satisfy all the conditions.

1.9 Methods of Integral Transforms

In the method of separation of variables our aim was to keep the independent variables in a PDE as separate as possible. We now discuss the use of integral transforms in solving PDEs, a method by which one of the independent variables can be eliminated from the differential coefficients.

The method consists simply of transforming the PDE into one containing derivatives with respect to a smaller number of variables [1,80]. Thus, if the original equation has just two independent variables, it may be possible to reduce the PDE into a soluble ODE. The solution obtained can then (where possible) be transformed back to give the solution of the original PDE. As we shall see, boundary conditions can usually be incorporated in a natural way.

Which sort of transform to use, and the choice of the variable (s) with respect to which the transform is to be taken, is a matter of experience; we illustrate this in the example below. In practice, transforms can be taken with respect to each variable in turn, and the transformation that affords the greatest simplification can be pursued further.

1.9-1 Laplace Transform

We have dealt with the Laplace transform when solving linear ODEs with constant coefficients, where the equations are transformed in this way to algebraic equations. This idea can be easily extended to PDEs, where the transformation leads to the decrease of the number of independent variables. PDEs in two variables are thus reduced to ODEs.

Let $u = u(t)$ be a piecewise continuous function on $[0, \infty)$ that “does not grow too fast”. Let us assume, for example, that u is of exponential order, which means that $|u(t)| \leq ce^{at}$ for t large enough, where $a, c > 0$ are appropriate constants. The Laplace transform of the function u is then defined by the formula

$$(\mathcal{L}u)(s) = U(s) = \int_0^{\infty} u(t) e^{-st} dt. \quad (1.53)$$

Here U and s are the transformed variables, U is the dependent one, s is the independent one, and U is defined for $s > a$ with $a > 0$, depending on $u(t)$. The function U is called the Laplace image of the function u , which is then called the original. The Laplace transform is a linear mapping, that is,

$$\mathcal{L}(c_1u + c_2v) = c_1\mathcal{L}u + c_2\mathcal{L}v,$$

where c_1, c_2 are arbitrary constants. If we know the Laplace image $U(s)$, then the original $u(t)$ can be obtained by the inverse Laplace transform of the image $U(s)$: $\mathcal{L}^{-1}U = u$. The

couples of Laplace images and their originals can be found in tables, or, in some cases, the transformation can be done using various software packages.

An important property of the Laplace transform, as well as of other integral transforms, is the fact that it turns differential operators in originals into multiplication operators in images. The following formulas hold [80]:

$$(\mathcal{L}u')(s) = sU(s) - u(0), \quad (1.54)$$

$$(\mathcal{L}u^{(n)})(s) = s^n U(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \dots - u^{(n-1)}(0), \quad (1.55)$$

if the derivatives considered are transformable (i.e., piecewise continuous functions of exponential order). To be precise, we should write $\lim_{t \rightarrow 0^+} u(t)$, $\lim_{t \rightarrow 0^+} u'(t)$, ... instead of

$u(0)$, $u'(0)$, However, without loss of generality, we can assume that the function u and its derivatives are continuous from the right at 0. Relations (1.54), (1.55) can be easily derived directly from the definition using integration by parts. Applying the Laplace transform to a linear ODE with constant coefficients, we obtain a linear algebraic equation for the unknown function $U(s)$. After solving it, we find the original function $u(t)$ by the inverse transform.

The same idea can be exploited also when solving PDEs for functions of two variables, say $u = u(x, t)$. The transformation will be done with respect to the time variable $t \geq 0$, the spatial variable x will be treated as a parameter unaffected by this transform. In particular, we define the Laplace transform of a function $u(x, t)$ by the formula

$$(\mathcal{L}u)(x, s) \equiv U(x, s) = \int_0^\infty u(x, t) e^{-st} dt. \quad (1.56)$$

The time derivatives are transformed in the same way as in the case of functions of one variable, that is, for example,

$$(\mathcal{L}u_t)(x, s) = sU(x, s) - u(x, 0).$$

The spatial derivatives remain unchanged, that is,

$$(\mathcal{L}u_x)(x, s) = \int_0^\infty \frac{\partial}{\partial x} u(x, t) e^{-st} dt = \frac{\partial}{\partial x} \int_0^\infty u(x, t) e^{-st} dt = U_x(x, s).$$

Thus, applying the Laplace transform to a PDE in two variables x and t , we obtain an ODE in the variable x and with the parameter s .

Example-1 (Constant Boundary Condition)

Using the Laplace transform, we solve the following initial boundary value problem for the diffusion equation. Let $u = u(x, t)$ denote the concentration of a chemical contaminant dissolved in a liquid on a half-infinite domain $x > 0$. Let us assume that, at time $t = 0$, the concentration is zero. On the boundary $x = 0$, constant-unit concentration of the contaminant is kept for $t > 0$. Assuming the unit diffusion constant, the behavior of the system is described by a mathematical model

$$\begin{aligned}
 u_t - u_{xx} &= 0, & x > 0, t > 0, \\
 u(x, 0) &= 0, \\
 u(0, t) &= 1, & u(x, t) \text{ bounded.}
 \end{aligned} \tag{1.57}$$

Here the boundedness assumption is related to the physical properties of the model and its solution. If we apply the Laplace transform to both sides of the equation, we obtain the following relation for the image U :

$$sU(x, s) - U_{xx}(x, s) = 0.$$

This is an ODE with respect to the variable x and with real positive parameter s . Its general solution has the form

$$U(x, s) = a(s)e^{-\sqrt{sx}} + b(s)e^{\sqrt{sx}}.$$

Since we require the solution u to be bounded in both variables x and t , the image U must be bounded in x as well. Thus, $b(s)$ must vanish, and hence

$$U(x, s) = a(s)e^{-\sqrt{sx}}.$$

Now, we apply the Laplace transform to the boundary condition obtaining $U(0, s) = \mathcal{L}(1) = 1/s$. It implies $a(s) = 1/s$ and the transformed solution has the form

$$U(x, s) = \frac{1}{s}e^{-\sqrt{sx}}.$$

Using the tables of the Laplace transform or some of the software packages, we easily find out that

$$u(x, t) = \operatorname{erfc}\left(\frac{x}{\sqrt{4t}}\right),$$

where erfc is the function defined by the relation [80]

$$\operatorname{erfc}(y) = 1 - \frac{2}{\sqrt{\pi}} \int_0^y e^{-r^2} dr = 1 - \operatorname{erf}(y).$$

In the previous example, we were able to find the original $u(x, t)$ to the Laplace image $U(x, s)$ using tables or software packages. There exists a general formula for inverse Laplace transform, which is based on theory of functions of complex variables [80]. It has a theoretical character, and from the practical point of view, it is used very rarely. In most cases, it is more or less useless.

1.9-2 Fourier Transform

The Fourier transform is another integral transform with properties similar to the Laplace transform. Since it again turns differentiation of the originals into multiplication of the images, it is a useful tool in solving differential equations. Contrary to the Laplace transform, which usually uses the time variable, the Fourier transform is applied to the spatial variable on the whole real line.

First, we start with functions of one spatial variable. The Fourier transform of a function $u = u(x), x \in \mathbb{R}$, is a mapping defined by the formula

$$(\mathcal{F}u)(\xi) \equiv \hat{u}(\xi) = \int_{-\infty}^{\infty} u(x)e^{-i\xi x} dx. \quad (1.58)$$

If $|u|$ is integrable in \mathbb{R} , that is, $\int_{-\infty}^{\infty} |u| dx < \infty$, then \hat{u} exists. However, the theory of the Fourier transform usually works with a smaller set of functions. We define the so called Schwartz space \mathcal{S} as the space of functions on \mathbb{R} that have continuous derivatives of all orders and that, together with their derivatives, decrease to zero for $x \rightarrow \pm\infty$ more rapidly than $|x|^{-n}$ for an arbitrary $n \in \mathbb{N}$. It means [80]

$$\mathcal{S} = \left\{ u \in C^\infty; \exists M = M(u) \in \mathbb{R}, \left| \frac{d^k u}{dx^k} \right| \leq M \frac{1}{|x|^n} \text{ for } |x| \rightarrow \infty, k \in \mathbb{N} \cup \{0\}; n \in \mathbb{N} \right\}.$$

It can be shown that, if $u \in \mathcal{S}$, then $\hat{u} \in \mathcal{S}$, and vice versa. We say that the Schwartz space \mathcal{S} is closed with respect to the Fourier transform.

It is important to mention that there exists no established convention how to define the Fourier transform. In literature, we can meet an equivalent of the definition formula (1.58) with the constant $1/\sqrt{2\pi}$ or $1/2\pi$ in front of the integral. There also exist definitions with positive sign in the exponent. We must keep this fact in mind while working with various sources or using the transformation tables.

The fundamental formula of the Fourier transform is that for the image of the k^{th} derivative $u^{(k)}$:

$$(\mathcal{F}u^{(k)})(\xi) = (i\xi)^k \hat{u}(\xi), \quad u \in \mathcal{S} \quad (1.59)$$

The derivation of this formula is based on integration by parts where all “boundary values” vanish due to zero values of the function and its derivatives at infinity. In the case of functions of two variables, say $u = u(x, t)$, the variable t plays the role of a parameter and we define

$$(\mathcal{F}u)(\xi, t) \equiv \hat{u}(\xi, t) = \int_{-\infty}^{\infty} u(x, t)e^{-i\xi x} dx. \quad (1.60)$$

The derivatives with respect to the spatial variable are transformed analogously as (1.59), the derivatives with respect to the time variable t stay unchanged; thus, for instance,

$$(\mathcal{F}u_x)(\xi, t) = (i\xi)\hat{u}(\xi, t),$$

$$(\mathcal{F}u_{xx})(\xi, t) = (i\xi)^2\hat{u}(\xi, t),$$

$$(\mathcal{F}u_t)(\xi, t) = \hat{u}_t(\xi, t).$$

The PDE in two variables x, t passes under the Fourier transform to an ODE in the t -variable. By its solving, we obtain the transformed function (the image) \hat{u} which can be converted to the original function u by the inverse Fourier transform

$$(\mathcal{F}^{-1}\hat{u})(x, t) \equiv u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi, t)e^{i\xi x} d\xi. \quad (1.61)$$

In comparison with the inverse Laplace transform, where the general inverse formula is quite complicated, this relation is very simple. Nevertheless, it is convenient to use the transformation tables or some software packages when solving particular problems.

Example-2 (Cauchy Problem for Wave Equation)

Let us solve the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, t > 0, \quad (1.62)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x).$$

We apply again the Fourier transform with respect to the spatial variable to the equation and both initial conditions. Thus, we obtain the transformed problem

$$\begin{aligned} \hat{u}_{tt}(\xi, t) + c^2 \xi^2 \hat{u}(\xi, t) &= 0, \\ \hat{u}(\xi, 0) &= \hat{\varphi}(\xi), \quad \hat{u}_t(\xi, 0) = \hat{\psi}(\xi). \end{aligned}$$

Its solution is the function

$$\hat{u}(\xi, t) = \hat{\varphi}(\xi) \cos c\xi t + \frac{1}{c\xi} \hat{\psi}(\xi) \sin c\xi t.$$

The solution of the original problem is then found by the inverse Fourier transform:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\hat{\varphi}(\xi) \cos c\xi t + \frac{1}{c\xi} \hat{\psi}(\xi) \sin c\xi t \right] e^{i\xi x} d\xi. \quad (1.63)$$

This integral expression, where, moreover, the Fourier transforms of the initial conditions occur, is not very transparent. Nevertheless, it can be converted to d'Alembert's formula [80]. Indeed, substituting the complex representation of the sine and cosine functions into (1.63), we obtain

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} \hat{\varphi}(\xi) (e^{ic\xi t} + e^{-ic\xi t}) e^{i\xi x} d\xi + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2ic\xi} \hat{\psi}(\xi) (e^{ic\xi t} - e^{-ic\xi t}) e^{i\xi x} d\xi. \quad (1.64)$$

The first integral on the right-hand side can be written as

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} (\hat{\varphi}(\xi) e^{i(x+ct)\xi} + \hat{\varphi}(\xi) e^{i(x-ct)\xi}) d\xi,$$

which is (using the definition of the inverse Fourier transform (1.61)) exactly the first term in d'Alembert's formula

$$\frac{1}{2} (\varphi(x + ct) + \varphi(x - ct)).$$

Similarly, the second integral term in (1.64) equals

$$\frac{1}{4\pi c} \int_{-\infty}^{\infty} \frac{1}{i\xi} \hat{\psi}(\xi) (e^{i(x+ct)\xi} - e^{i(x-ct)\xi}) d\xi = \frac{1}{4\pi c} \int_{-\infty}^{\infty} \hat{\psi}(\xi) \int_{x-ct}^{x+ct} e^{iy\xi} dy d\xi.$$

Changing the order of integration and using again the inverse Fourier transform, we obtain the second term in d'Alembert's formula

$$\frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy.$$

Remark-1

In some cases, the methods of integral transforms are applicable also to equations with non-constant coefficients. Let us consider, for example, the Cauchy problem for the transport equation

$$\begin{aligned} tu_x + u_t &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= f(x). \end{aligned}$$

Since the varying coefficient is-in this case-the time variable t , we use the Fourier transform with time playing the role of a parameter. We have

$$\mathcal{F}(tu_x) = t\mathcal{F}(u_x)i\xi t\hat{u}.$$

Transforming the equation and the initial conditions, we obtain

$$i\xi t\hat{u} + \hat{u}_t = 0, \quad \hat{u}(\xi, 0) = \hat{f}(\xi),$$

and hence

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-i\frac{t^2}{2}\xi}.$$

By the inverse Fourier transform (e.g., using the transformation formulas), we obtain the solution of the original equation in the form

$$u(x, t) = f\left(x - \frac{t^2}{2}\right).$$

Remark-2

The Laplace and Poisson equations can also be solved, in some cases, by the method of integral transforms. As an example, let us consider the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u(x, 0) &= f(x), \\ u(x, y) &\text{ bounded for } y \rightarrow \infty \end{aligned}$$

We will search for a solution using the Fourier transform with respect to x . Its application to our problem leads to the equation

$$\hat{u}_{yy} - \xi^2\hat{u} = 0,$$

whose general solution is $\hat{u}(\xi, y) = a(\xi)e^{-\xi y} + b(\xi)e^{\xi y}$. The boundedness assumption implies

$$b(\xi) = 0 \quad \text{for } \xi > 0,$$

$$a(\xi) = 0 \quad \text{for } \xi > 0.$$

Hence, $\hat{u}(\xi, y) = c(\xi)e^{-|\xi|y}$. Here, a, b, c are arbitrary functions. If we take into account the boundary condition, we derive $c(\xi) = \hat{f}(\xi)$ and thus

$$\hat{u}(\xi, y) = e^{-|\xi|y} \hat{f}(\xi).$$

The inverse transformation leads to the solution of the original problem in the form of a convolution:

$$u(x, t) = \left(\frac{y}{\pi} \frac{1}{x^2 + y^2} \right) * f = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)\tau}{(x-\tau)^2 + y^2}.$$

It is noticed that in this convolution y is just a parameter.

1.10 Methods of Green's Functions

1.10-1 Green's Function

Green's functions are named after the mathematician and physicist George Green who was born in Nottingham in 1793 and "invented" the Green's function in 1828. This invention was developed in an essay written by Green entitled "Mathematical Analysis to the Theories of Electricity and Magnetism" originally published in Nottingham in 1828 and reprinted by the George Green Memorial Committee to mark the bicentenary of the birth of George Green in 1993 [81]. In this essay, Green's function solutions to the Laplace and Poisson equation are formulated (but not in the manner considered in this section, in which the Green's function is defined using the delta function).

The Green's function is possibly one of the most powerful tools for solving partial differential equations - a tool that is all the more enigmatic in that the work of George Green was neglected for nearly 30 years after his death in 1841 and to this day no one knows what he looked like or how and why he developed his revolutionary ideas.

The Green's function was successfully applied to classical electromagnetism and acoustics in the late 19th century. More recently, the Green's function has been the working tool of calculations in particle physics, condensed matter and solid-state physics, quantum mechanics and many other topics of advanced applied mathematics and mathematical physics. Just as the Green's function revolutionized classical field theory in the 19th century (hydrodynamics, electrostatics, and magnetism) so it revolutionized quantum field theory in the mid-20th century through the introduction of quantum Green's functions. This provided the essential link between the theories of quantum electrodynamics in the 1940s and 1950s and has played a major role in theoretical physics ever since. It is interesting to note that the pioneering work of Richard Feynman in the 1950s and 1960s which led to the

development of the Feynman diagram was based on the Green's function; in fact, the Feynman diagram can be considered to be a pictorial representation of a Green's function (a Green's function associated with wave operators) - what Feynman referred to as a "propagator" [81].

Accordingly, the method of Green's functions is an important technique for solving boundary value, initial and boundary value, and Cauchy problems for partial differential equations. It is most commonly identified with the solution of boundary value problems for Laplace's equation and a Green's function has already been introduced in that context. We begin by constructing generalizations of Green's second theorem that are appropriate for the second order differential equations introduced in the present chapter. These integral theorems are then used to show how boundary value, initial and boundary value, and Cauchy problems can be solved in terms of appropriately defined Green's functions for each of these problems. Even though the construction of Green's functions requires that a problem like the original (given) problem must be solved, it is often easier to solve the Green's function problem in a number of important cases as we shall see. In this regard the fundamental solutions, of which Green's functions are a special case, play an important role. Since the determination and use of Green's functions require the use of generalized functions such as the Dirac delta function, a brief discussion of the theory of generalized functions will be given. Most of the discussion, however, is devoted to the construction and use of Green's functions for problems involving equations of elliptic, hyperbolic, and parabolic types.

Green's functions are used mainly to solve certain types of linear inhomogeneous partial differential equations (although homogeneous partial differential equations can also be solved using this approach). In principle, the Green's function technique can be applied to any linear constant coefficient inhomogeneous partial differential equation (either scalar or vector) in any number of independent variables although in practice difficulties can arise in computing the Green's function analytically. In fact, Green's functions provide more than just a solution; it could be said that they transform a partial differential equation representation of a physical problem into an integral equation representation - the kernel of the integral equation being composed (completely or partly) of the Green's function associated with the partial differential equation. This is why Green's function solutions are considered to be one of the most powerful analytical tools we have for solving partial differential equations - equations that arise in areas such as electromagnetism (Maxwell's equations), wave mechanics (elastic wave equation), optics (Helmholtz equation), quantum mechanics (Schrodinger and Dirac equations), fluid dynamics (viscous fluid equation), relativistic particle dynamics (Klein-Gordon equation), general relativity (Einstein equations) to name but a few.

1.10-2 Integral Theorems and Green's Functions

In this section we construct integral theorems appropriate for the elliptic hyperbolic, and parabolic equations introduced in this chapter. Each of these theorems follows from an application of the divergence theorem and represents a generalization of Green's second theorem. These theorems form the basis for the construction of the Green's functions we consider in this section. Technically, the theorems are valid only if the functions occurring in the integrals are sufficiently smooth and this is generally not the case for Green's

functions. Nevertheless, we shall assume these theorems are formally valid in all cases and rely on the theory of generalized functions to form a basis for their validity, even though this is not demonstrated.

By way of a short introduction we now consider two short examples. The first example is based on considering point sources to generate a solution to an ordinary differential equation and is based on a "qualitative analysis". The second example makes specific use of the delta function and its properties to develop a solution which is based on a more systematic analysis.

Example-1

Consider the following inhomogeneous ordinary differential equation

$$\hat{L}u(x) = f(x), \quad (1.65)$$

where \hat{L} is a linear differential operator and $f(x)$ is a given function (the source term), the solution being required on the interval $0 \leq x \leq a$ where a is some constant. Instead of considering $f(x)$ as a continuous source function, let us approximate it by a set of discrete source functions $f(\xi_1), f(\xi_2), \dots, f(\xi_n)$ acting at the points $x = \xi_1, x = \xi_2, \dots, x = \xi_n$ all for $x \in [0, a]$. Now, define the function $g(x; \xi_i)$ to be the solution to equation (1.65) due to a point source acting at ξ_i . The solution due to the single effect of this point source is $g(x; \xi_i)f(\xi_i)$. The solution for $u(x)$ is then obtained by summing the results for all the n -point source terms acting over the interval $0 \leq x \leq a$ and takes the form

$$u(x) = \sum_{i=1}^n g(x; \xi_i)f(\xi_i).$$

As n becomes larger, so that the number of point source $f(\xi_i)$ increases, a better approximation to $f(x)$ is obtained. In the limit as $n \rightarrow \infty$, so $|\xi_i - \xi_{i+1}| \rightarrow 0 \forall i$ the summation in the equation above may be replaced by an integral to give the required solution to equation (1.65) in the form

$$u(x) = \int_0^a g(x; \xi)f(\xi)d\xi.$$

The function $g(x; \xi)$ is called the Green's function of the problem. The notation used to write a Green's function changes considerable from author to author. They are usually written in the form $g(x; \xi)$ (as in this example) or $g(x|\xi)$, both forms actually being equivalent to $g(|x - \xi|)$ as we shall see later. Similar results to the one given above may be obtained for linear partial differential equations: for example, the solution of the Poisson equation in two dimensions.

$$\nabla^2 u(x, y) = f(x, y); \quad x \in [0, a], \quad y \in [0, b]$$

may be written as

$$u(x, y) = \int_0^a \int_0^b g(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta,$$

where $g(x, y; \xi, \eta)$ is the Green's function of the problem. The approach for developing a Green's function solution discussed in this example is based on considering point sources

to provide a set of elementary results and then summing up the results to give the required solution. In optics and acoustics, this principle is often referred to as Huygens' principle which allows the optical or acoustic field generated by a given source to be computed by considering the field generated from a single point on the source and then summing up the field generated from a large collection of such points. In this sense, the principle behind a Green's function solution is effectively the same as Huygens' principle, i.e. find the solution to the problem for a single point and then integrate over all such points. The relationship between a point source which is described mathematically by the delta function and the Green's function is an important one.

By way of a short introduction to the use of the delta function for solving partial differential equations using Green's functions, we consider the following example which in comparison with the first example, provides a more complete form of analysis to develop a Green's function solution for the one-dimensional inhomogeneous wave equation.

Example-2

Consider the equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)u(x, k) = f(x) \quad (1.66)$$

where k (the wave number) is a constant and $f(x)$ is the source term, the solution being required over all space $x \in (-\infty, \infty)$ subject to the conditions that u and $\partial u/\partial x$ are zero at $\pm\infty$. This equation describes the behavior of "steady waves" (constant wavelength $\lambda = 2\pi/k$) due to a source $f(x)$.

Define the Green's function as being the solution to the equation obtained by replacing the source term with a delta function which represents a point source at x_0 say, giving the equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)g(x|x_0, k) = \delta(x - x_0) \quad (1.67)$$

where δ has the following fundamental property

$$\int_{-\infty}^{\infty} u(x)\delta(x - x_0)dx = u(x_0).$$

Multiplying equation (1.66) by g gives

$$g\left(\frac{\partial^2}{\partial x^2} + k^2\right)u = gf$$

and multiplying equation (1.67) by u gives

$$u\left(\frac{\partial^2}{\partial x^2} + k^2\right)g = u\delta(x - x_0).$$

Now subtract the two results and integrate to obtain

$$\int_{-\infty}^{\infty} \left(g\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 g}{\partial x^2}\right)dx = \int_{-\infty}^{\infty} fgdx - \int_{-\infty}^{\infty} u\delta(x - x_0)dx.$$

Using the generalized sampling property of the delta function given above and rearranging the result, we obtain

$$u(x_0, k) = \int_{-\infty}^{\infty} f g dx - \int_{-\infty}^{\infty} \left(g \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 g}{\partial x^2} \right) dx .$$

Evaluating the second integral on the right-hand side,

$$\begin{aligned} \int_{-\infty}^{\infty} \left(g \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 g}{\partial x^2} \right) dx &= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x} \left(g \frac{\partial u}{\partial x} \right) - \frac{\partial g}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left(u \frac{\partial g}{\partial x} \right) + \frac{\partial u}{\partial x} \frac{\partial g}{\partial x} \right] dx \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(g \frac{\partial u}{\partial x} \right) dx - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(u \frac{\partial g}{\partial x} \right) dx = \left[g \frac{\partial u}{\partial x} \right]_{-\infty}^{\infty} - \left[u \frac{\partial g}{\partial x} \right]_{-\infty}^{\infty} = 0, \end{aligned}$$

provided u and $\partial u / \partial x$ are zero at $x = \pm \infty$. With these conditions, we obtain the Green's function solution to equation (1.66) in the form

$$u(x_0, k) = \int_{-\infty}^{\infty} f(x) g(x|x_0, k) dx.$$

Note, that this result is a “type” of convolution - the convolution of the source function f with the Green's function g . Physically, Green's functions associated with wave type problems as in this example, represents the way in which a wave propagates from one point in space to another. For this reason, they are sometimes referred to as propagators. In this case, the Green's function is a function of the “path length” between x and x_0 irrespective of whether $x > x_0$ or $x < x_0$. The path length is given by $|x - x_0|$ and the Green's function is a function of this path length which is why, using the notation $x|x_0 \equiv |x - x_0|$, we write $g(x|x_0)$.

The solution given above is of little value unless the Green's function can be computed. Before the computational techniques associated with Green's functions can be studied, it is necessary to be familiar with a class of functions known as generalized functions and in particular, one of the most commonly used generalized function, the delta function.

1.10-3 Green's Functions for the Time-independent Wave Equation

In this section, we shall concentrate on the computation of Green's functions for the time-independent wave equation in one, two and three dimensions. The solution is over all space and the Green's function is not constrained to any particular boundary conditions (except those at $\pm \infty$). It is therefore referred to as a free space Green's function.

(i) The One-dimensional Green's Function

We start by reconsidering Example-2 given in Section 1.10-2 which, through the application of the sampling property of the delta function together with some relatively simple analysis, demonstrated that the solution to the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) u(x, k) = f(x),$$

for constant k and $x \in (-\infty, \infty)$ subject to the boundary conditions

$$u(x, k)|_{\pm\infty} = 0 \quad \text{and} \quad \left[\frac{\partial}{\partial x} u(x, k) \right]_{\pm\infty} = 0,$$

is given by

$$u(x_0, k) = \int_{-\infty}^{\infty} f(x) g(x|x_0, k) dx,$$

where g is the Green's function. This solution is of course worthless without an expression for the Green's function which is given by the solution to the equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) g(x|x_0, k) = -\delta(x - x_0), \quad (1.68)$$

subject to $g(x|x_0, k)|_{\pm\infty} = 0$ and $[\partial g(x|x_0, k)/\partial x]_{\pm\infty} = 0$. We shall therefore start by looking at the evaluation of the Green's function for this case. Note that in this case, the Green's function is defined for $-\delta$ on the right-hand side instead of $+\delta$. This is for convenience only in the computations which follow; it does not affect the analysis and is merely a convention which ultimately reduces the number of negative signs associated with the calculation. For this reason, many authors define the Green's function with $-\delta$, a definition which may also be used.

The solution to this equation is based on employing the properties of the Fourier transform. Writing $X = |x - x_0|$, we express g and δ as Fourier transforms, that is

$$g(X, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(u, k) \exp(iuX) du, \quad (1.69)$$

and

$$\delta(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iuX) du.$$

Substituting these expressions into equation (1.68) and differentiating gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (-u^2 + k^2) G(u, k) \exp(iuX) du = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iuX) du,$$

from which it follows that

$$G(u, k) = \frac{1}{u^2 - k^2}.$$

Substituting this result back into equation (1.69), we obtain

$$g(X, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(iuX)}{u^2 - k^2} du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(iuX)}{(u - k)(u + k)} du.$$

The problem is therefore reduced to that of evaluating the above integral. This can be done using Cauchy's integral formula [82],

$$\oint_C f(z) dz = 2\pi i \times (\text{sum of the residues enclosed by } C),$$

where C is the contour defining the path of integration. In order to evaluate the integral explicitly using this formula, we must consider the singular nature or poles of the integrand at $z = -k$ and $z = k$. For now let us consider a contour which encloses both poles.

The residue at $z = k$ is given by $\exp(ikX)/(2k)$ and at $z = -k$ by $-\exp(-ikX)/(2k)$. Hence, the Green's function is given by

$$g(X, k) = 2\pi i \left(\frac{\exp(ikX)}{4\pi k} - \frac{\exp(-ikX)}{4\pi k} \right) = -\frac{1}{k} \sin(kX).$$

This Green's function represents the propagation of waves travelling away from the point disturbance at $x = x_0$ or "outgoing waves" and also waves travelling towards the point disturbance or "incoming waves". Since x and x_0 are points along a line, we can consider the result to be the sum of waves travelling to the left of $\delta(x - x_0)$ in which $x < x_0$ and to the right of $\delta(x - x_0)$ where $x > x_0$. In most applications it is convenient to consider the Green's function for outgoing or (more rarely) incoming waves but not both. Here, the Green's function for incoming waves is given by

$$g(x|x_0, k) = -\frac{i}{2k} \exp(-ik|x - x_0|),$$

and for outgoing waves is

$$g(x|x_0, k) = \frac{i}{2k} \exp(ik|x - x_0|).$$

(ii) The Two-dimensional Green's Function

In two dimensions, the same method can be used to obtain the (free space) Green's function [82], that is to solve the equation

$$(\nabla^2 + k^2)g(\mathbf{r}|\mathbf{r}_0, k) = -\delta^2(\mathbf{r} - \mathbf{r}_0)$$

subject to some boundary conditions at $|\mathbf{r}| = \infty$, where

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y; \quad \mathbf{r}_0 = \hat{\mathbf{x}}x_0 + \hat{\mathbf{y}}y_0$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Note that

$$\delta^2(\mathbf{r} - \mathbf{r}_0) \equiv \delta(x - x_0)\delta(y - y_0).$$

Also note that g is a function of the path length $|\mathbf{r} - \mathbf{r}_0|$. Writing $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ and using the same technique as before, namely the one used to derive an integral representation of the one-dimensional Green's function, we obtain

$$g(R, k) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\exp(i\mathbf{u}\cdot\mathbf{R})}{u^2 - k^2} d^2\mathbf{u}.$$

In polar coordinates this result becomes

$$g(R, k) = \frac{1}{(2\pi)^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{\exp(iuR \cos \theta)}{u^2 - k^2} u du d\theta.$$

Integrating over u first and using Cauchy's residue theorem, we have

$$\oint_C \frac{z \exp(izR \cos \theta)}{(z+k)(z-k)} dz = i\pi \exp(ikR \cos \theta),$$

where the contour of integration C has been chosen to enclose just one of the poles at $z = k$. This provides an expression for the "outgoing" Green's function in which the wave propagates away from the point disturbance at \mathbf{r}_0 .

A solution for the pole at $z = -k$ would provide a solution which represents a wave-field converging on \mathbf{r}_0 . The "outgoing" Green's function is usually the most physically significant result (accept for an implosion for example). Thus, the (outgoing) Green's function can be written in the form

$$g(R, k) = \frac{i}{4k} \int_0^\pi \exp(ikR \cos \theta) d\theta.$$

Writing the Green's function in this form allows us to employ the result

$$H_0^{(1)}(kR) = \frac{i}{\pi} \int_0^\pi \exp(ikR \cos \theta) d\theta,$$

where $H_0^{(1)}$ is the Hankel function (of the first kind and of order zero). This is the integral representation for the Hankel transform and it can be used to write the two-dimensional Green's function as

$$g(\mathbf{r}|\mathbf{r}_0, k) = \frac{i}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_0|).$$

A useful form of this function can be obtained by employing the asymptotic approximation

$$H_0^{(1)}(kR) \simeq \sqrt{\frac{2}{\pi}} \exp(-i\pi/4) \frac{\exp(-ikR)}{\sqrt{kR}},$$

which is valid when

$$kR \gg 1.$$

This condition means that the wavelength of the wave originating from \mathbf{r}_0 is very small compared with the distance between \mathbf{r}_0 and \mathbf{r} which is physically reasonable in many cases and so a two-dimensional Green's function of the following form can be used:

$$g(\mathbf{r}|\mathbf{r}_0, k) = \frac{1}{\sqrt{8\pi}} \exp(i\pi/4) \frac{\exp(ik|\mathbf{r} - \mathbf{r}_0|)}{\sqrt{k|\mathbf{r} - \mathbf{r}_0|}}.$$

(iii) The Three-dimensional Green's Function

In three-dimensions, the free space Green's function is given by the solution to the equation

$$(\nabla^2 + k^2)g(\mathbf{r}|\mathbf{r}_0, k) = -\delta^3(\mathbf{r} - \mathbf{r}_0),$$

with boundary conditions at $|\mathbf{r}| = \infty$, where

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z; \quad \mathbf{r}_0 = \hat{\mathbf{x}}x_0 + \hat{\mathbf{y}}y_0 + \hat{\mathbf{z}}z_0;$$

$$\delta^3(\mathbf{r} - \mathbf{r}_0) \equiv \delta(x - x_0)\delta(y - y_0)\delta(z - z_0),$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In this case

$$g(R, k) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\exp(i\mathbf{u} \cdot \mathbf{R})}{u^2 - k^2} d^3\mathbf{u}.$$

CHAPTER - 2

NUMERICAL METHODS OF SOLUTIONS OF SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

2.1 Introduction

Many problems in mathematics, science and engineering are not simple and cannot be solved by exact closed-form analytical formulas. It is often necessary to obtain approximate numerical or asymptotic solutions rather than exact solutions. Many numerical methods that have evolved over the years reduce algebraic or differential equations to discrete form which can be solved easily by computer. However, if the numerical method is not carefully chosen, the numerically computed solution may not be anywhere close to the true solution. Another problem is that the computation for a difficult problem may take so long that it is impractical for a computer to carry out [1].

Partial differential equations are involved in the description of virtually every physical situation where quantities vary in space or in space and time. These include phenomena as diverse as diffusion, electromagnetic waves, hydrodynamics, and quantum mechanics (Schrödinger waves). In all but the simplest cases, these equations cannot be solved analytically, and so numerical methods must be employed for quantitative results. In a typical numerical treatment, the dependent variables (such as temperature or electrical potential) are described by their values at discrete points (a lattice) of the independent variables (e.g., space and time) and, by appropriate discretization, the partial differential equation is reduced to a large set of difference equations. Although these difference equations then can be solved, in principle, by the direct matrix methods, the large size of the matrices involved (dimension comparable to the number of lattice points, often more than several thousand) makes such an approach impractical. Fortunately, the locality of the original equations (i.e., they involve only low-order derivatives of the dependent variables) makes the resulting difference equations “sparse” in the sense that most of the elements of the matrices involved vanish. For such matrices, iterative methods of inversion and diagonalization can be very efficient.

An important aspect of numerical analysis of partial differential equations is the numerical solution of the finite linear algebraic systems that are generated by the discrete equations. These are in general very large, but with sparse matrices, which makes iterative methods suitable. The development of convergence analysis for such methods has paralleled that of the error analysis sketched above. In the 1950s and 1960s particular attention was paid to systems associated with finite difference approximation of positive type of second-order elliptic equations, particularly the five-point scheme, and starting with the Jacobi and Gauss-Seidel methods techniques were developed such as the Frankel and Young successive over relaxation and the Peaceman-Rachford (1955) alternating direction methods, as described in the influential book of Varga [83]. In later years, systems with positive-definite matrices stemming from finite element methods have been solved first by the conjugate gradient method proposed by Hestenes and Stiefel (1952), and then making

this more effective by preconditioning. The multigrid method was first introduced for finite difference methods in the 1960s by Fedorenko and Bahvalov and further developed by Brandt in the 1970s. For finite elements the multigrid method and the associated method of domain decomposition have been and are being intensely pursued by, e.g., Braess, Hackbusch, Bramble, and Widlund.

In this section we attempt to give a personal account of the development of numerical analysis of partial differential equations. We begin with the introduction in the 1930s and further development of the finite difference method and then describe the subsequent appearance around 1960 and increasing role of the finite element method. Even though clearly some ideas may be traced back further, our starting point will be the fundamental theoretical paper by Courant, Friedrichs and Lewy (1928) on the solution of problems of mathematical physics by means of finite differences. In that paper a discrete analogue of Dirichlet's principle was used to define an approximate solution by means of the five-point approximation of Laplace's equation, and convergence as the mesh width tends to zero was established by compactness. A finite difference approximation was also defined for the wave equation, and the CFL stability condition was shown to be necessary for convergence; again, compactness was used to demonstrate convergence. Since the purpose was to prove existence of solutions, no error estimates or convergence rates were derived. With its use of a variational principle for discretization and its discovery of the importance of mesh-ratio conditions in approximation of time-dependent problems that paper points forward and has had a great influence on numerical analysis of partial differential equations.

Error bounds for difference approximations of elliptic problems were first derived by Gerschgorin (1930) whose work was based on a discrete analogue of the maximum principle for Laplace's equation. This approach was actively pursued through the 1960s by, e.g., Collatz, Motzkin, Wasow, Bramble, and Hubbard, and various approximations of elliptic equations and associated boundary conditions were analyzed.

For time-dependent problems considerable progress in finite difference methods was made during the period of, and immediately following, the Second World War, when large-scale practical applications became possible with the aid of computers. A major role was played by the work of von Neumann, partly reported in O'Brien, Hyman and Kaplan (1951). For parabolic equations a highlight of the early theory was the important paper by John (1952). For mixed initial-boundary value problems the use of implicit methods was also established in this period by, e.g., Crank and Nicolson (1947). The finite difference theory for general initial value problems and parabolic problems then had an intense period of development during the 1950s and 1960s, when the concept of stability was explored in the Lax equivalence theorem and the Kreiss matrix lemmas, with further major contributions given by Douglas, Lees, Samarskii, Widlund and others. For hyperbolic equations, and particularly for nonlinear conservation laws, the finite difference method has continued to play a dominating role up until the present time, starting with work by, e.g., Friedrichs, Lax, and Wendroff. Standard references on finite difference methods are the textbooks of Collatz [84], Forsythe and Wasow [85] and Richtmyer and Morton [86].

The idea of using a variational formulation of a boundary value problem for its numerical solution goes back to Lord Rayleigh (1894, 1896) and Ritz (1908), see, e.g., Kantorovich

and Krylov [87]. In Ritz's approach the approximate solution was sought as a finite linear combination of functions such as, for instance, polynomials or trigonometrical polynomials. The use in this context of continuous piecewise linear approximating functions based on triangulations adapted to the geometry of the domain was proposed by Courant (1943) in a paper based on an address delivered to the American Mathematical Society in 1941. Even though this idea had appeared earlier, also in work by Courant himself (see Babuska [88]), this is often thought of as the starting point of the finite element method, but the further development and analysis of the method would occur much later. The idea to use an orthogonality condition rather than the minimization of a quadratic functional is attributed to Galerkin (1915); its use for time-dependent problems is sometimes referred to as the Faedo-Galerkin method, cf. Faedo (1949), or, when the orthogonality is with respect to a different space, as the Petrov- Galerkin or Bubnov-Galerkin method.

As a computational method the finite element method originated in the engineering literature, where in the mid-1950s structural engineers had connected the well-established framework analysis with variational methods in continuum mechanics into a discretization method in which a structure is thought of as divided into elements with locally defined strains or stresses. Some of the pioneering work was done by Turner, Clough, Martin and Topp (1956) and Argyris [89] and the name of the finite element method appeared first in Clough (1960). The method was later applied to other classes of problems in continuum mechanics; a standard reference from the engineering literature is Zienkiewicz [90].

Independently of the engineering applications a number of papers appeared in the mathematical literature in the mid-1960s which were concerned with the construction and analysis of finite difference schemes by the Rayleigh-Ritz procedure with piecewise linear approximating functions, by, e.g., Oganessian (1962, 1966), Friedrichs (1962), Cea (1964), Demjanovic (1964), Feng (1965), and Friedrichs and Keller (1966) (who considered the Neumann problem). Although, in fact, special cases of the finite element method, the methods studied were conceived as finite difference methods; they were referred to in the Russian literature as variational difference schemes.

In the period following this, the finite element method with piecewise polynomial approximating functions was analyzed mathematically in work such as Birkhoff, Schultz and Varga (1968), in which the theory of splines was brought to bear on the development, and Zlamal (1968), with the first stringent a priori error analysis of more complicated finite elements. The so called mixed finite element methods, which are based on variational formulations where, e.g., the solution of an elliptic equation and its gradient appear as separate variables and where the combined variable is a saddle point of a Lagrangian functional, were introduced in Brezzi (1974); such methods have many applications in fluid dynamical problems and for higher-order elliptic equations.

Following Babuska (1976), Babuska and Rheinboldt (1978), much effort has been devoted to showing a posteriori error bounds which depend only on the data and the computed solution. Such error bounds can be applied to formulate adaptive algorithms which are of great importance in computational practice. Comprehensive references for the analysis of the finite element method are Babuska and Aziz [91], Strang and Fix [92], Ciarlet [93], and Brenner and Scott [94].

Simultaneous with this development other classes of methods have arisen which are related to the above, and we will sketch four such classes: In a collocation method an approximation is sought in a finite element space by requiring the differential equation to be satisfied exactly at a finite number of collocation points, rather than by an orthogonality condition. In a spectral method one uses globally defined functions, such as eigenfunctions, rather than piecewise polynomials approximating functions, and the discrete solution may be determined by either orthogonality or collocation. A finite volume method applies to differential equations in divergence form. Integrating over an arbitrary volume and transforming the integral of the divergence into an integral of a flux over the boundary, the method is based on approximating such a boundary integral. In a boundary integral method, a boundary value problem for a homogeneous elliptic equation in a J -dimensional domain is reduced to an integral equation on its $d - 1$ -dimensional boundary, which in turn can be solved by, e.g., the Galerkin finite element method or by collocation.

Many ideas and techniques are common to the finite difference and the finite element methods, and in some simple cases they coincide. Nevertheless, with its more systematic use of the variational approach, its greater geometric flexibility, and the way it more easily lends itself to error analysis, the finite element method has become the dominating approach both among numerical analysts and in applications. The growing need for understanding the partial differential equations modeling the physical problems has seen an increase in the use of mathematical theory and techniques and has attracted the interest of many mathematicians. The computer revolution has made large-scale real-world problems accessible to simulation, and in later years the concept of computational mathematics has emerged with a somewhat broader scope than classical numerical analysis.

The most commonly used numerical methods are finite differences that give pointwise approximations of the governing equations. These methods can be used successfully to solve many difficult problems, but their major weakness is that they are not suitable for problems with irregular geometries, curved boundaries or unusual boundary conditions. For example, the finite difference methods are not particularly effective for a circular domain because a circle cannot be accurately partitioned into rectangles. However, there are other numerical methods including the finite element method and the boundary element method. Unlike finite difference methods, the finite element method can be used effectively to determine fairly accurate approximate solutions to a wide variety of governing equations defined over irregular regions. The entire solution domain can be modeled analytically or approximated by replacing it with small, interconnected discrete finite elements (hence the name finite element). The solution is then approximated by extremely simple functions (linear functions) on these small elements such as triangles. These small elements are collected together, and requirements of continuity and equilibrium are satisfied between neighboring elements.

In a nutshell, the basic idea of the finite element method (FEM) consists of decomposing a given domain into a set of finite elements of arbitrary shape and size. This decomposition is usually called a mesh or a grid with the restriction that elements cannot overlap nor leave any part of the domain uncovered. For each element, a certain number of points is introduced that can be located on the edges of the elements or inside. These points are

Let $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ be the initial approximation to the solution of system (2.2). Substituting these values into system (2.2), we get the residuals

$$\begin{aligned} R_1^{(0)} &= c_1 - x_1^{(0)} + \sum_{j=2}^n b_{1j} x_j^{(0)}, \\ R_2^{(0)} &= c_2 - x_2^{(0)} + \sum_{\substack{j=1 \\ j \neq 2}}^n b_{2j} x_j^{(0)}, \\ &\dots\dots\dots \\ R_n^{(0)} &= c_n - x_n^{(0)} + \sum_{j=1}^{n-1} b_{nj} x_j^{(0)} \end{aligned} \quad (2.3)$$

If we give an increment of $\delta x_s^{(0)}$ to one of the unknowns $x_s^{(0)}$, then the corresponding residual $R_s^{(0)}$ will be diminished by the quantity $\delta x_s^{(0)}$, and all the other residuals $R_i^{(0)}$ ($i \neq s$) will be increased by the quantity $b_{is} \delta x_s^{(0)}$. Thus, to make the next residual $R_s^{(1)}$ vanish, it suffices to give the quantity $x_s^{(0)}$ an increment of

$$\delta x_s^{(0)} = R_s^{(0)}$$

and we have

$$R_s^{(1)} = 0$$

and

$$R_i^{(1)} = R_i^{(0)} + b_{is} \delta x_s^{(0)} \quad \text{for } i \neq s$$

The method of relaxation [98,99] in its simplest form consists in reducing the numerically largest residual to zero at each step by changing the value of the appropriate component of the approximation. The process is terminated when all the residuals of the last transformed system are equal to zero within the required accuracy. We will not consider the question of the convergence of this process [99].

Example

Solve the following system by the method of relaxation [*]:

$$\begin{aligned} 10x_1 - 2x_2 - 2x_3 &= 6, \\ -x_1 + 10x_2 - 2x_3 &= 7, \end{aligned} \quad (2.4)$$

$$-x_1 - x_2 + 10x_3 = 8$$

carrying the computations to two decimal places.

Solution

We reduce the system (2.4) to a form convenient for relaxation:

$$-x_1 + 0.2x_2 + 0.2x_3 + 0.6 = 0,$$

$$0.1x_1 - x_2 + 0.2x_3 + 0.7 = 0,$$

$$0.1x_1 + 0.1x_2 - x_3 + 0.8 = 0$$

choosing the zero values

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$$

for the initial approximations of the roots, we get the respective residuals:

$$R_1^{(0)} = 0.60, \quad R_2^{(0)} = 0.70, \quad R_3^{(0)} = 0.80$$

By the general theory, we assume

$$\delta x_3^{(0)} = 0.80$$

whence we get the residuals

$$R_1^{(1)} = R_1^{(0)} + 0.2 \times 0.8 = 0.60 + 0.16 = 0.76,$$

$$R_2^{(1)} = R_2^{(0)} + 0.2 \times 0.8 = 0.70 + 0.16 = 0.86,$$

$$R_3^{(1)} = R_3^{(0)} - R_2^{(0)} = 0$$

Now we set

$$\delta x_2^{(1)} = 0.86$$

and so on. The result of the computations are given in Table 1.

Summing all the increments $\delta_i^{(k)}$ ($i = 1, 2, 3; k = 0, 1, \dots$), we get the values of the roots:

$$x_1 = 0 + 0.93 + 0.07 = 1.00,$$

$$x_2 = 0 + 0.86 + 0.13 + 0.01 = 1.00,$$

$$x_3 = 0 + 0.80 + 0.18 + 0.02 = 1.00$$

Check by substituting the values of the roots thus found into the original equations; in this case the system (2.4) has been solved exactly.

Table-2.1 Solution of a linear system by the method of relaxation

	x_1	R_1	x_2	R_2	x_3	R_3
	0	0.60	0	0.70	0	0.80
		0.16		0.16	0.80	-0.80
		0.76		0.86		0
		0.17	0.86	-0.86		0.09
		0.93		0		0.09
	0.93	-0.93		0.09		0.09
		0		0.09		0.18
		0.04		0.04	0.18	-0.18
		0.04		0.13		0
		0.03		-0.13		0.01
		0.07	0.13	0		0.01
		-0.07		0.01		0.01
	0.07	0		0.01		0.02
		0		0		-0.02
		0		0.01	0.02	0
		0		-0.01		0
		0	0.01	0		0
Σ	1.00		1.00		1.00	

2.3 Discretization and the Variational Principle

Most of the physically important partial differential equations are of second order and can be classified into three types: elliptic, parabolic, and hyperbolic. Elliptic partial differential equations involve second order derivatives in each of the independent variables, each derivative having the same sign when all terms in the equation are grouped on one side. We will consider in the following discussion particular form of elliptic boundary value problem for a field in two spatial dimensions (x, y) . The boundary value problem involves the equation

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = S(x, y). \quad (2.5)$$

Although this is not the most general elliptic form, it nevertheless covers a wide variety of situations [100]. For example, in an electrostatic problem, φ is the potential and S is

related to the charge density, while in a steady-state heat diffusion problem, φ is the temperature, and S is the local rate of heat generation or loss. Equation (2.5) by itself is an ill-posed problem, as some sort of boundary conditions is required [101]. These may be of the Dirichlet type [102] for the stream function and the vorticity, and of the Neumann type [102] for the pressure, in problems of fluid mechanics. Accordingly, φ is specified on some large closed curve in the (x, y) plane (conveniently, the unit square). The boundary value problem is then to use (2.5) to find φ everywhere within the unit square.

Other classes of boundary value problems, such as Neumann (where the normal derivative of φ is specified on the surfaces) and mixed type (where a linear combination of φ and its normal derivative is specified), can be handled by very similar methods.

The eigenvalue problems we will be interested in might involve an equation of the form

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi + V(x, y)\varphi = \varepsilon\varphi, \quad (2.6)$$

together with a set of Dirichlet boundary conditions. This might arise, for example, as the time-independent Schrödinger equation, where φ is the wave function, V proportional to the potential, and ε is related to the eigenvalue. Such an equation might also be used to describe the fields in an acoustic or electromagnetic waveguide, where ε is then related to the square of the cut-off frequency. The eigenvalue problem then consists of finding the values ε_λ and the associated eigenfunctions φ_λ for which equation (2.6) and the boundary conditions are satisfied. As methods for solving such problems are closely related to those for solving a related parabolic equation we will defer their discussion later.

The ultimate goal of the field of computational fluid dynamics is to understand the physical events that occur in the flow of fluids around and within designated objects. These events are related to the action and interaction of phenomena such as dissipation, diffusion, convection, shock waves, slip surfaces, boundary layers, and turbulence. In the field of aerodynamics, all of these phenomena are governed by the compressible Navier-Stokes equations. Many of the most important aspects of these relations are nonlinear and, as a consequence, often have no analytic solution. This, of course, motivates the numerical solution of the associated partial differential equations. The use of numerical methods to solve partial differential equations introduces an approximation that, in effect, can change the form of the basic partial differential equations themselves. The new equations, which are the ones actually being solved by the numerical process, are often referred to as the modified partial differential equations. Since they are not precisely the same as the original equations, they can, and probably will, simulate the physical phenomena listed above in ways that are not exactly the same as an exact solution to the basic partial differential equation [102]. Mathematically, these differences are usually referred to as truncation errors. However, the theory associated with the numerical analysis of fluid mechanics was developed predominantly by scientists deeply interested in the physics of fluid flow and, therefore, these errors are often identified with a particular physical phenomenon on which they have a strong effect. Thus, methods are said to have a lot of artificial viscosity or said to be highly dispersive. This means that the errors caused by the numerical approximation result in a modified partial differential equation having additional terms that can be

identified with the physics of dissipation in the first case and dispersion in the second. There is nothing wrong, of course, with

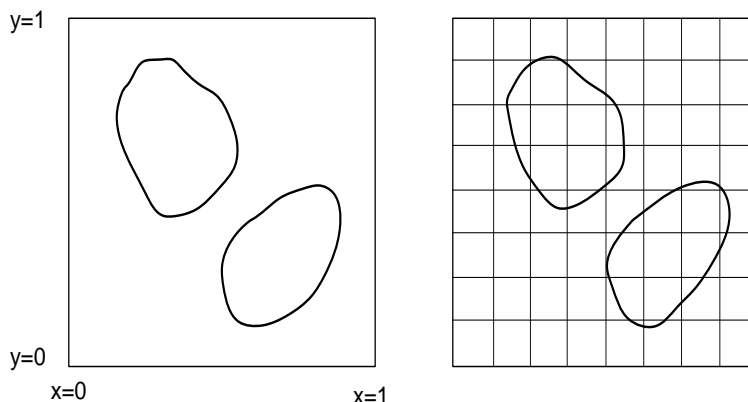


Figure-2.1 (Left) A two-dimensional boundary value problem with Dirichlet boundary conditions. Values of φ are specified on the edges of the unit square and on the surfaces within. (Right) Discretization of the problem on a uniform Cartesian lattice.

identifying an error with a physical process, nor with deliberately directing an error to a specific physical process, as long as the error remains in some engineering sense small. However, if used and interpreted properly, these methods give very useful information.

Our first step is to cast Eq. (2.5) in a form suitable for numerical treatment. To do so, we define a lattice of points covering the region of interest in the (x, y) plane. For convenience, we take the lattice spacing, h , to be uniform and equal in both directions, so that the unit square is covered by $(N + 1) \times (N + 1)$ lattice points (see Figure-2.1). These points can be labeled by indices (i, j) , each of which runs from 0 to N , so that the coordinates of the point (i, j) are $(x_i = ih, y_j = jh)$. If we then define $\varphi_{ij} = \varphi(x_i, y_j)$, and similarly for S_{ij} , it is then straightforward to apply the three-point difference approximation for the second derivative in each direction and so approximate (2.5) as

$$-\left[\frac{\varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{ij}}{h^2} + \frac{\varphi_{i,j+1} + \varphi_{i,j-1} - 2\varphi_{ij}}{h^2} \right] = S_{ij} \tag{2.7}$$

or, in a more convenient notation,

$$-\left[\delta_i^2 \varphi_{ij} + \delta_j^2 \varphi_{ij} \right] = h^2 S_{ij} \tag{2.8}$$

Here, δ_i^2 is the second-difference operator in the i index,

$$\delta_i^2(\varphi_{ij}) = \varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{ij}$$

and δ_j^2 is defined similarly.

Although equations (2.7), (2.8) are the equations we will be solving, it is useful to derive it in a different way, based on a variational principle. Such an approach is handy in cases

where the coordinates are not Cartesian, or when more accurate difference formulas are required. It is also guaranteed to lead to symmetric (or Hermitian) difference equations, an often-useful property. The variational method also affords some insight into how the solution algorithm works. A good review of this approach can be found in [103].

Consider the quantity E , defined to be a functional of the field φ of the form [100]

$$E = \int_0^1 dx \int_0^1 dy \left[\frac{1}{2} (\nabla\varphi)^2 - S\varphi \right] \quad (2.9)$$

In some situations, E has a physical interpretation. For example, in electrostatics, $-\nabla\varphi$ is the electric field and S is the charge density, so that E is indeed the total energy of the system. However, in other situations, such as the steady-state diffusion equation, E should be viewed simply as a useful quantity.

It is easy to show that, at a solution to (2.5), E is stationary under all variations $\delta\varphi$ that respect the Dirichlet boundary conditions imposed. Indeed, the variation is

$$\delta E = \int_0^1 dx \int_0^1 dy [\nabla\varphi \cdot \nabla\delta\varphi - S\delta\varphi] \quad (2.10)$$

which upon integrating the second derivative term by parts becomes.

$$\delta E = \int_C dl \delta\varphi \mathbf{n} \cdot \nabla\varphi + \int_0^1 dx \int_0^1 dy \delta\varphi [-\nabla^2\varphi - S] \quad (2.11)$$

where the line integral is over the boundary of the region of interest (C) and \mathbf{n} is the unit vector normal to the boundary. Since we consider only variations that respect the boundary conditions, $\delta\varphi$ vanishes on C , so that the line integral does as well. Demanding that δE be zero for all such variations then implies that φ satisfies (2.5). This then furnishes a variational principle for our boundary value problem.

To derive a discrete approximation to the partial differential equation based on this variational principle, we first approximate E in terms of the values of the field at the lattice points and then vary with respect to them. The simplest approximation to E is to employ the two-point difference formula to approximate each first derivative in $(\nabla\varphi)^2$ at the points halfway between the lattice points and to use the trapezoidal rule for the integrals. This leads to [100]

$$E = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \left[(\varphi_{ij} - \varphi_{i-1,j})^2 + (\varphi_{ij} - \varphi_{i,j-1})^2 \right] - h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} S_{ij} \varphi_{ij} \quad (2.12)$$

putting

$$\frac{\partial E}{\partial \varphi_{ij}} = 0$$

for all ij then leads to the difference equation derived previously, (2.11), (2.12). Of course, a more accurate discretization can be obtained by using better approximations for the first derivatives and for the integrals, taking care that the accuracies of both are commensurate. It is also easy to show from (2.12) that not only is E stationary at the solution, but that it is a minimum as well.

We must now discuss where the boundary conditions enter the set of linear equations (2.7), (2.8). Unless the coordinate system is well adapted to the geometry of the surfaces on which the boundary conditions are imposed (e.g., the surfaces are straight lines in Cartesian coordinates or arcs in cylindrical or spherical coordinates), the lattice points will only roughly describe the geometry (see Figure-2.1). One can always improve the accuracy by using a non-uniform lattice spacing and placing more points in the regions near the surfaces or by transforming to a coordinate system in which the boundary conditions are expressed more naturally. In any event, the boundary conditions will then provide the values of the φ_{ij} at some subset of lattice points. At a point far away from one of the boundaries, the boundary conditions do not enter (2.7), (2.8) directly. However, consider (2.7), (2.8) at a point just next to a boundary, say $(i, N - 1)$. Since φ_{iN} is specified as part of the boundary conditions (as it is on the whole border of the unit square), we can rewrite (2.8) as

$$4\varphi_{i,N-1} - \varphi_{i+1,N-1} - \varphi_{i-1,N-1} - \varphi_{i,N-2} = h^2 S_{i,N-1} + \varphi_{iN}; \quad (2.13)$$

that is φ_{iN} enters not as an unknown, but rather as an inhomogeneous, known term. Similarly, if a Neumann boundary condition were imposed at a surface, say $\frac{\partial \varphi}{\partial y} = g(x)$ at $y = 1$ or, equivalently, $j = N$, then this could be approximated by the discrete boundary condition.

$$\varphi_{iN} - \varphi_{i,N-1} = hg_i$$

which means that at $j = N - 1$, equation (2.7) would become

$$3\varphi_{i,N-1} - \varphi_{i+1,N-1} - \varphi_{i-1,N-1} - \varphi_{i,N-2} = h^2 S_{i,N-1} + hg_i \quad (2.14)$$

These considerations, and a bit more thought, show that the discrete approximation to the differential equation (2.5) is equivalent to a system of linear equations for the unknown values of φ at the interior points. In a matrix notation, this can be written as [100]

$$M\varphi = S, \quad (2.15)$$

where M is the matrix appearing in the linear system (2.7), (2.8) and the inhomogeneous term s is proportional to S at the interior points and is linearly related to the specified values of φ or its derivatives on the boundaries. In any sort of practical situation there are a very large number of these equations (some 2500 if $N = 50$, say), so that solution by direct inversion of M is impractical. Fortunately, since the discrete approximation to the Laplacian involves only neighboring points, most of the elements of M vanish (it is sparse) and there are then efficient iterative techniques for solving (2.15). We begin their discussion by considering an analogous, but simpler, one-dimensional boundary value problem, and then return to the two-dimensional case.

2.4 An Iterative Method for Boundary Value Problems

The one-dimensional boundary value problem analogous to the two-dimensional problem we have been discussing can be written as

$$-\frac{d^2\varphi}{dx^2} = S(x), \quad (2.16)$$

with $\varphi(0)$ and $\varphi(1)$ specified. The related variational principle involves [100]

$$E = \int_0^1 dx \left[\left(\frac{d\varphi}{dx} \right)^2 - S\varphi \right], \quad (2.17)$$

Which can be discretized on a uniform lattice of spacing $h = 1/N$ as

$$E = \frac{1}{2h} \sum_{i=1}^N (\varphi_i - \varphi_{i-1})^2 - h \sum_{i=1}^{N-1} S_i \varphi_i, \quad (2.18)$$

when varied with respect to φ_i , this yields the difference equation.

$$2\varphi_i - \varphi_{i+1} - \varphi_{i-1} = h^2 S_i, \quad (2.19)$$

which is, of course, just the naive discretization of equation. (2.16)

Methods of solving the boundary value problem by integrating forward and backward in x may also be discussed, but we can also consider (2.19), together with the known values of φ_0 and φ_N as a set of linear equations. For a modest number of points (say $N < 100$), the linear system above can be solved by the direct methods and, in fact, a very efficient special direct method exists for such “tri-diagonal” systems. However, to illustrate the iterative methods appropriate for the large sparse matrices of elliptic partial differential equations in two or more dimensions, we begin by rewriting (2.19) in a “solved” form for φ_i :

$$\varphi_i = \frac{1}{2} [\varphi_{i+1} + \varphi_{i-1} + h^2 S_i]. \quad (2.20)$$

Although this equation is not manifestly useful, since we don't know the φ, S on the right-hand side, it can be interpreted as giving us an “improved” value for φ_i based on the values of φ at the neighboring points. Hence the strategy (Gauss-Seidel iteration) is to guess some initial solution and then to sweep systematically through the lattice (say from left to right), successively replacing φ at each point by an improved value. Note that the most “current” values of the $\varphi_{i\pm 1}$ are to be used in the right-hand side of equation (2.20). By repeating this sweep many times, an initial guess for φ can be “relaxed” to the correct solution.

To investigate the convergence of this procedure, we generalize equation (2.20) so that at each step of the relaxation φ_i is replaced by a linear mixture of its old value and the “improved” one given by (2.20):

$$\varphi_i \rightarrow \varphi'_i = (1 - \omega)\varphi_i + \frac{1}{2}\omega[\varphi_{i+1} + \varphi_{i-1} + h^2 S_i] \quad (2.21)$$

Here, ω is a parameter that can be adjusted to control the rate of relaxation: “over-relaxation” corresponds to $\omega > 1$, while “underrelaxation” means $\omega < 1$. The optimal value of ω that maximizes the rate of relaxation will be discussed below. To see that (2.21) results in an “improvement” in the solution, we calculate the change in the energy functional (2.18), remembering that all φ 's except φ_i are to be held fixed. After some algebra, one finds [100]

$$E' - E = -\frac{\omega(2-\omega)}{2h} \left[\frac{1}{2}(\varphi_{i+1} + \varphi_{i-1} + h^2 S_i) - \varphi_i \right]^2 \leq 0, \quad (2.22)$$

so that, as long as $0 < \omega < 2$, the energy never increases, and should thus converge to the required minimum value as the sweeps proceed. (The existence of other, spurious minima of the energy would imply that the linear system (2.19) is not well posed.)

As an example of this relaxation method, let us consider the one-dimensional boundary-value problem of the form (2.16) with

$$S(x) = 12x^2; \quad \varphi(0) = \varphi(1) = 0.$$

The exact solution is $\varphi(x) = x(1 - x^3)$.

And the energy is $E = -\frac{9}{14} = -0.64286$.

Results for the energy as a function of iteration number are shown in Table 2.1 for three different values of ω . [100]

Table-2.2 Convergence of the energy functional during relaxation of a 1-D boundary value problem

Iteration	$\omega = 0.5$	$\omega = 1.0$	$\omega = 1.5$
1	-0.01943	-0.04959	-0.09459
21	-0.24267	-0.44024	-0.60688
41	-0.36297	-0.56343	-0.63700
61	-0.44207	-0.61036	-0.63631
81	-0.49732	-0.62795	-0.63836
101	-0.53678	-0.63450	-0.63837
121	-0.56517	-0.63693	-0.63837
141	-0.58563	-0.63783	-0.63837
161	-0.60037	-0.63817	-0.63837
181	-0.61100	-0.63829	-0.63837
201	-0.61866	-0.63834	-0.63837
221	-0.62418	-0.63836	-0.63837
241	-0.62815	-0.63836	-0.63837
----	-----	-----	-----
381	-0.63734	-0.63837	-0.63837
401	-0.63763	-0.63837	-0.63837

Despite the rather poor initial guess for φ , the iterations converge, and the converged energy is independent of the relaxation parameter but differs somewhat from the exact answer due to discretization errors (i.e., h not vanishingly small); the discrepancy can be reduced, of course, by increasing N . A detailed examination of the solution indicates good agreement with the analytical result. Note that the rate of convergence clearly depends upon ω . A general analysis [104] shows that the best choice for the relaxation parameter depends upon the lattice size and on the geometry of the problem; it is usually greater than 1. The optimal value can be determined empirically by examining the convergence of the solution for only a few iterations before choosing a value to be used for many iterations.

The application of the relaxation scheme described above to two- (or even three-) dimensional problems is now straightforward. Upon solving (2.7) for φ_{ij} we can generate the analogue of (2.17) [100]:

$$\varphi_{ij} \rightarrow \varphi'_{ij} = (1 - \omega)\varphi_{ij} + \frac{\omega}{4} [\varphi_{i+1,j} + \varphi_{i,j+1} + \varphi_{i,j-1} + h^2 S_{ij}] \quad (2.23)$$

If this algorithm is applied successively to each point in the lattice, say sweeping the rows in order from top to bottom and each row from left to right, one can show that the energy functional (2.12) always decreases (if ω is within the proper range) and that there will be convergence to the required solution.

Several considerations can serve to enhance this convergence in practice. First, starting from a good guess at the solution (perhaps one with similar, but simpler, boundary conditions) will reduce the number of iterations required. Second, an optimal value of the relaxation parameter should be used, either estimated analytically or determined empirically, as described above. Third, it may sometimes be more efficient to concentrate the relaxation process, for several iterations, in some sub-area of the lattice where the trial solution is known to be particularly poor, thus not wasting effort on already-relaxed parts of the solution. Finally, one can always do a calculation on a relatively coarse lattice that relaxes with a small amount of numerical work, and then interpolate the solution found onto a finer lattice to be used as the starting guess for further iterations.

2.5 More on Discretization

It is often the case that the energy functional defining a physical problem has a form more complicated than the simple “integral of the square of the derivative” that we have been considering so far. For example, in an electrostatics problem with spatially varying dielectric properties or in a diffusion problem with a spatially-varying diffusion coefficient, the boundary-value problem (2.5) is modified to

$$-\nabla \cdot D\nabla\varphi = S(x, y), \quad (2.24)$$

where $D(x, y)$ is the dielectric constant or diffusion coefficient, and the corresponding energy functional is (compare equation (2.9))

$$E = \int_0^1 dx \int_0^1 dy \left[\frac{1}{2} D(\nabla\varphi)^2 - S\varphi \right]. \quad (2.25)$$

Although it is possible to discretize equation (2.24) directly, it should be evident from the previous discussion that a far better procedure is to discretize (2.25) first and then to differentiate with respect to the field variables to obtain the difference equations to be solved.

To see how this works out in detail, consider the analog of the one-dimensional problem defined by (2.17),

$$E = \int_0^1 dx \left[\frac{1}{2} D(x) \left(\frac{d\varphi}{dy} \right)^2 - S\varphi \right], \quad (2.26)$$

The discretization analogous to (2.18) is [100]

$$E = \frac{1}{2h} \sum_{i=1}^N D_{i-\frac{1}{2}} (\varphi_i - \varphi_{i-1})^2 - h \sum_{i=1}^{N-1} S_i \varphi_i, \quad (2.27)$$

where $D_{i-\frac{1}{2}}$ is the diffusion constant at the half-lattice points. This might be known directly if we have an explicit formula for $D(x)$, or it might be approximated with appropriate accuracy by $\frac{1}{2} (D_i + D_{i-1})$. Note that, in either case, we have taken care to center the differencing properly. Variation of this equation then leads directly to the corresponding difference equations (compare equation (2.19)),

$$\left(D_{i+\frac{1}{2}} + D_{i-\frac{1}{2}} \right) \varphi_i - D_{i-\frac{1}{2}} \varphi_{i+1} - D_{i-\frac{1}{2}} \varphi_{i-1} = h^2 S_i. \quad (2.28)$$

These can then be solved straightforwardly by the relaxation technique described above.

A problem treated in cylindrical or spherical coordinates presents very much the same kind of situation. For example, when the diffusion or dielectric properties are independent of space, the energy functional in cylindrical coordinates will involve [100]

$$E = \int_0^\infty dr r \left[\frac{1}{2} \left(\frac{d\varphi}{dr} \right)^2 - S\varphi \right], \quad (2.29)$$

where r is the cylindrical radius. (We suppress here the integrations over the other coordinates.) This is of the form (2.26), with $D(r) = r$ and an additional factor of r appearing in the source integral. Discretization on a lattice $r_i = h_i$ in analogy to (2.27) then leads to the analog of (2.28),

$$2r_i \varphi_i - r_{i+\frac{1}{2}} \varphi_{i+1} - r_{i-\frac{1}{2}} \varphi_{i-1} = h^2 r_i S_i \quad (2.30)$$

At $i = 0$, this equation just tells us that $\varphi_1 = \varphi_{-1}$, or equivalently that $\frac{\partial \varphi}{\partial r} = 0$ at $r = 0$. This is to be expected, as, in the electrostatics language, Gauss' law allows no radial electric field at $r = 0$.

At $i = 1$, equation (2.30) gives an equation involving three unknowns φ_0 , φ_1 , and φ_2 , but putting $\varphi_0 = \varphi_1$ as a rough approximation to the zero derivative boundary condition gives an equation involving only two unknowns, which is what we expect at a boundary on the basis of our experience with the Cartesian problems discussed above.

A more elegant discretization of problems with cylindrical symmetry naturally incorporates the zero-derivative boundary condition at $r = 0$ by working on a lattice defined by $r_i = \left(1 - \frac{1}{2}\right)h$. In this case, equation (2.30) is still valid, but for $i = 1$ the coefficient of the term involving φ_{t-1} vanishes, giving directly an equation with only two unknowns, φ_1 and φ_2 .

2.6 The Finite Element Method

In order to develop the finite element method [1], we recall Euler–Lagrange equation

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) = 0, \quad a \leq x \leq b. \quad (2.31)$$

with $u(a) = \alpha$ and $u(b) = \beta$. We divide the interval $a \leq x \leq b$ into n parts by the R_{n+1} set: $a = x_0 < x_1 < x_2 < \dots < x_n = b$. Each such subinterval is called an element. In general, the length of the elements need not be equal, though for simplicity, we assume that they are equal in length so that $h = \frac{1}{n}(b - a)$. We set $u_k = u(x_k)$, $k = 0, 1, 2, \dots, n$ so that $u_0(x_0) = \alpha$ and $u_n(x_n) = \beta$, while u_1, u_2, \dots, u_{n-1} are unknown quantities. We next rewrite the functional

$$I(u) = \int_a^b F(x, u, u') dx, \quad u' = \frac{du}{dx}, \quad (2.32)$$

as

$$I(u) = \int_{x_0}^{x_1} F(x, u, u') dx + \int_{x_1}^{x_2} F(x, u, u') dx + \dots + \int_{x_{n-1}}^{x_n} F(x, u, u') dx. \quad (2.33)$$

We define a piecewise linear interpolating function $L(x)$ of u_i as the function which is continuous on $[a, b]$ and whose graph consists of straight line segments joining the consecutive pairs of points (x_k, u_k) , (x_{k+1}, u_{k+1}) for $k = 0, 1, 2, \dots, (n - 1)$, that is,

$$L(x) = u_k + \frac{1}{h}(u_{k+1} - u_k)(x - x_k), \quad x_k \leq x \leq x_{k+1}, \quad (2.34)$$

where $k = 0, 1, 2, \dots, (n - 1)$.

Substituting L for u and L' for u' in (2.33) and assuming that the integrals can be computed exactly yields

$$I_{n-1} = I_{n-1}(u_1, u_2, \dots, u_{n-1}). \quad (2.35)$$

We next find the minimum of I_{n-1} by solving the system of equations

$$\frac{\partial I_{n-1}}{\partial u_k} = 0, \quad k = 0, 1, 2, \dots, (n - 1). \quad (2.36)$$

The solution of this system (2.36) is then substituted into (2.34) to obtain a continuous, piecewise linear approximation for the exact solution $u(x)$.

Example 1. (The Dirichlet problem for the Poisson equation in a plane).

We consider the problem

$$-\nabla^2 u = f(x, y) \text{ in } D, \tag{2.37}$$

$$u = 0 \text{ on the boundary } \partial D. \tag{2.38}$$

The region D is first triangulated so that it is approximated by a region D_n which is the union of a finite number of triangles as shown in Figures-2.2(a) and 2.2(b). We denote the interior vertices by V_1, V_2, \dots, V_n .

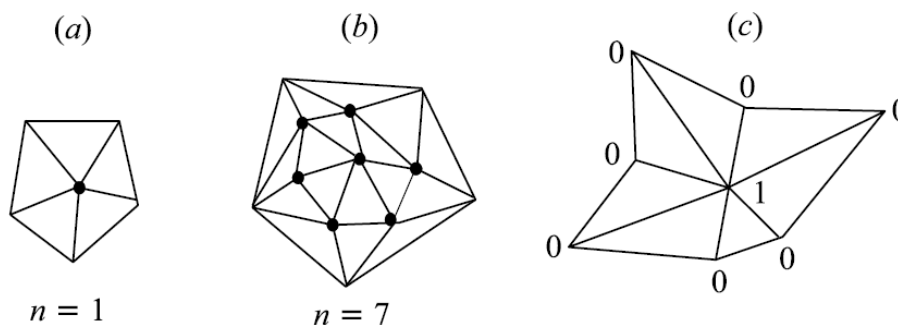


Figure-2.2 (a), (b), and (c). Triangular elements.

We next choose n trial functions $v_1(x, y), v_2(x, y), \dots, v_n(x, y)$, one for each interior vertex. Each trial function $v_m(x, y)$ is assumed to be equal to 1 at its vertices V_m and equal to zero at all other vertices as in shown in Figure-2.2(c).

Each linear trial function $v_m(x, y) = ax + by + c$, where a, b , and c are different for each trial function and for each triangle. This requirement determines $v_m(x, y)$ uniquely. Indeed, its graph is simply a pyramid of unit height with its peak at V_m and it is zero on all the triangles which do not touch V_m .

We next approximate the solution $u(x, y)$ by a linear combination of the $v_m(x, y)$ so that

$$u_n(x, y) = a_1 v_1(x, y) + a_2 v_2(x, y) + \dots + a_n v_n(x, y) = \sum_{m=1}^n a_m v_m(x, y), \tag{2.39}$$

where the coefficients a_1, a_2, \dots, a_n are to be determined.

We multiply the Poisson equation (2.37) by any function $v(x, y)$ which is zero on ∂D and next use Green's first identity to obtain

$$\iint_D \nabla u \cdot \nabla v dx dy = \iint_D f v dx dy. \tag{2.40}$$

We assume that (2.40) is valid only for the first n trial functions so that $v = v_k$ for $k = 0, 1, 2, \dots, n$. With $u(x, y) = u_n(x, y)$ and $v(x, y) = v_k(x, y)$, result (2.40) becomes

$$\sum_{m=1}^n a_m \left[\iint_D (\nabla u_m \cdot \nabla v_k) dx dy \right] = \iint_D f v_k dx dy \quad (2.41)$$

This is a system of n linear equations, where $m = 1, 2, \dots, n$ in n unknown coefficients a_m , and can be rewritten in the form

$$\sum_{m=1}^n \alpha_{mk} a_m = f_k, \quad k = 1, 2, \dots, n \quad (2.42)$$

where

$$\alpha_{mk} = \iint_D (\nabla u_m \cdot \nabla v_k) dx dy, \quad f_k = \iint_D f v_k dx dy. \quad (2.43)$$

Consequently, the finite element method leads to finding α_{mk} and f_k from (2.43) and then solving (2.42). Finally, the approximate value of the solution $u(x, y)$ is then given by (2.39).

Several comments are in order. First, the trial functions $v_m(x, y)$ depend on the geometry of the problem and are completely known. Second, the approximate solution $u_n(x, y)$ vanishes on the boundary ∂D_n . Third, at a vertex $V_i = (x_i, y_i)$,

$$u_n(x_i, y_i) = a_i v_i(x_i, y_i) + \dots + a_n v_n(x_i, y_i) = \sum_{r=1}^n a_r v_r(x_i, y_i) = a_i,$$

where

$$v_r(x_k, y_k) = \begin{cases} 0, & r \neq k \\ 1, & r = k. \end{cases}$$

Fourth, the coefficients a_i are exactly the values of the approximate solution at the vertices $V_i = (x_i, y_i)$.

Example-2

We consider the variational problem of finding the extremes of the functional

$$I(u) = \int_0^6 (u'^2 + u^2 - 2u - 2xu) dx, \quad (2.44)$$

with the boundary conditions $u(0) = 1$ and $u(6) = 7$.

We divide $0 \leq x \leq 6$ into three equal parts of length $h = 2$ by $x_0 = 0$, $x_1 = 2$, $x_2 = 4$, and $x_3 = 6$. We set $u_k = u(x_k)$ so that $u_0 = u(0) = 1$ and $u_3 = u(6) = 7$, while u_1 and u_2 are unknown quantities. We have

$$F(x, u, u') = u'^2 + u^2 - 2u - 2xu \quad (2.45)$$

so that

$$I = \int_0^2 F(x, u, u') dx + \int_2^4 F(x, u, u') dx + \int_4^6 F(x, u, u') dx \quad (2.46)$$

We take

$$L(x) = \begin{cases} u_0 + \frac{1}{2}(u_1 - u_0)x, & 0 \leq x \leq 2 \\ u_1 + \frac{1}{2}(u_2 - u_1)(x - 2), & 2 \leq x \leq 4, \\ u_2 + \frac{1}{2}(u_3 - u_1)(x - 4), & 4 \leq x \leq 6, \end{cases} \quad (2.47)$$

so that its derivative is

$$L'(x) = \begin{cases} \frac{1}{2}(u_1 - u_0), & 0 \leq x \leq 2 \\ \frac{1}{2}(u_2 - u_1), & 2 \leq x \leq 4 \\ \frac{1}{2}(u_3 - u_1), & 4 \leq x \leq 6. \end{cases} \quad (2.48)$$

Substituting (2.47) and (2.48) into (2.46) and using (2.45) we get

$$I = \int_0^2 \left[\left(\frac{u_1 - u_0}{2} \right)^2 + \left\{ u_0 + \frac{1}{2}(u_1 - u_0)x \right\}^2 - 2 \left\{ u_0 + \frac{1}{2}(u_1 - u_0)x \right\} - 2x \left\{ u_0 + \frac{1}{2}(u_1 - u_0)x \right\} \right] dx + \int_2^4 \left[\left(\frac{u_2 - u_1}{2} \right)^2 + \left\{ u_1 + \frac{1}{2}(u_2 - u_1)(x - 2) \right\}^2 - 2 \left\{ u_1 + \frac{1}{2}(u_2 - u_1)(x - 2) \right\} - 2x \left\{ u_1 + \frac{1}{2}(u_2 - u_1)(x - 2) \right\} \right] dx + \int_4^6 \left[\left(\frac{u_3 - u_1}{2} \right)^2 + \left\{ u_2 + \frac{1}{2}(u_3 - u_1)(x - 4) \right\}^2 - 2 \left\{ u_2 + \frac{1}{2}(u_3 - u_1)(x - 4) \right\} - 2x \left\{ u_2 + \frac{1}{2}(u_3 - u_1)(x - 4) \right\} \right] dx.$$

Using the known values of u_0 and u_3 and integrating we obtain ($n = 3$)

$$I_2 = \frac{7}{3}(u_1^2 + u_2^2) - \frac{1}{3}u_1u_2 - \frac{37}{3}u_1 - \frac{67}{3}u_2 - \frac{101}{3}.$$

Consequently, equation (2.36) gives two equations

$$\frac{\partial I_2}{\partial u_1} = \frac{14}{3}u_1 - \frac{1}{3}u_2 - \frac{37}{3} = 0,$$

$$\frac{\partial I_2}{\partial u_2} = -\frac{1}{3}u_1 + \frac{14}{3}u_2 - \frac{67}{3} = 0.$$

Thus, the solutions for u_1 and u_2 are $u_1 = 3$ and $u_2 = 5$.

Putting these values into (2.47) leads to the approximate solution

$$L(x) = 1 + x, \quad 0 \leq x \leq 6. \quad (2.49)$$

In this problem, the Euler–Lagrange equation for (2.44) is given by

$$u'' - u = -(1 + x). \quad (2.50)$$

Solving this equation with $u(0) = 1$ and $u(6) = 7$ yields the exact solution

$$u(x) = 1 + x.$$

In this example, the exact and approximate solutions are identical due to the simplicity of the problem. In general, these solutions will be different.

We close this section by adding some comments on another numerical technique known as the boundary element method (boundary integral equation method). This method was widely used in early research in solid mechanics, fluid mechanics, potential theory and electromagnetic theory. However, the major breakthrough in the boundary integral equation method came in 1963 when two classic papers were published by Jaswon (1963) [105] and Symm (1963) [106]. The boundary element method is based on the mathematical aspect of finding the Green's function solution of differential equations with prescribed boundary conditions. It also uses Green's theorem to reduce a volume problem to a surface problem, and a surface problem to a line problem. This technique is not only very useful but also very accurate for linear problems, especially for three dimensional problems with rapidly changing variables in fracture and contact problems in solid mechanics. However, this method is computationally less efficient than the finite element method and is not widely used in industry. It is fairly popular for finding numerical solutions of acoustic problems. Since the early 1970s the boundary element method has continued to develop at a fast pace and has been extended to include a wide variety of linear and nonlinear problems in continuum mechanics.

2.7 The Boundary Element Methods

Over recent decades, the boundary element method (BEM) has received much attention from researchers and has become an important technique in the computational solution of a number of physical problems. In common with the better-known finite element method (FEM) and finite difference method (FDM), the boundary element method is essentially a method for solving partial differential equations (PDEs) and can only be employed when the physical problem can be expressed as such. As with the other methods mentioned, the boundary element method is a numerical method and hence it is an important subject of research amongst the numerical analysis community. However, the potential advantages of the BEM have seemed so considerable that the strongest impetus behind its development has come from the engineering community, in its enthusiasm to obtain flexible and efficient computer-based solutions to a range of engineering problems. The boundary element method has found application in such diverse topics as stress analysis, potential flow, fracture mechanics and acoustics.

The boundary element method is derived through the discretization of an integral equation that is mathematically equivalent to the original partial differential equation. The essential re-formulation of the PDE that underlies the BEM consists of an integral equation that is defined on the boundary of the domain and an integral that relates the boundary solution to the solution at points in the domain. The former is termed a boundary integral equation (BIE) and the BEM is often referred to as the boundary integral equation method or boundary integral method. Over the years the term boundary element method has become more popular [107]. The other terms are still used in the literature however, particularly when authors wish to refer to the overall derivation and analysis of the methods, rather than their implementation or application.

An integral equation re-formulation can only be derived for certain classes of PDE. Hence, the BEM is not widely applicable when compared to the near-universal adaptability

of the finite element and finite difference method. However, in the cases in which the boundary element method is applicable, it often results in a numerical method that is easier to use and more computationally efficient than the competing methods.

The advantage in the boundary element method arises from the fact that only the boundary (or boundaries) of the domain of the PDE requires sub-division. (In the finite element method or finite difference method the whole domain of the PDE requires discretization.) Thus the dimension of the problem is effectively reduced by one, for example an equation governing a three-dimensional region is transformed into one over its surface. In cases where the domain is exterior to the boundary, as it is in acoustic radiation and scattering models, the extent of the domain is infinite and hence the advantages of the BEM are even more striking; the equation governing the infinite domain is reduced to an equation over the (finite) boundary [107].

In a boundary integral method, a boundary value problem for a homogeneous PDE in a domain Ω with the solution u given on the boundary Γ is reformulated as an integral equation over Γ . This equation may then be used as a basis for numerical approximation. We shall illustrate this approach for the model problem [108]

$$\nabla^2 u = 0 \text{ in } \Omega \subset \mathbb{R}^2, \text{ with } u = g \text{ on } \Gamma, \quad (2.51)$$

where we assume that Γ is smooth. To pose the boundary integral equation, let

$$U(x) = -\frac{1}{2\pi} \ln|x|,$$

be the fundamental solution of the Laplacian in \mathbb{R}^2 . For any u with $\nabla^2 u = 0$ on Γ we have by Green's formula

$$u(x) = \int_{\Gamma} U(x-y) \frac{\partial u}{\partial n_y}(y) ds_y - \int_{\Gamma} \frac{\partial U}{\partial n_y}(x-y) u(y) ds_y, \quad x \in \Omega. \quad (2.52)$$

With x on Γ the integrals on the right define the single- and double-layer potentials $V \partial u / \partial n$ and Wu . We note that although $\nabla U(x-y)$ has a singularity of order $O(|x-y|^{-1})$, the kernel $(\partial U / \partial n_y)(x-y)$ is bounded for $x, y \in \Gamma$, so that the operator W is well defined. For $x \in \Omega$ approaching Γ the two integrals tend to $V \partial u / \partial n$ and $\frac{1}{2}u + Wu$, respectively, so that (2.52) yields

$$\frac{1}{2}u = V \partial u / \partial n - Wu \text{ on } \Gamma.$$

With $u = g$ on Γ this is a Fredholm integral equation of the first kind for the determination of $\partial u / \partial n$ on Γ , which inserted into (2.52) together with $u = g$ on Γ gives the solution of (2.51). Instead of this direct method one may use the indirect method of assuming that the solution of (2.52) may be represented as a potential of a function on Γ , so that

$$u(x) = \int_{\Gamma} \Phi(x-y)v(y) ds_y \quad \text{or} \quad u(x) = \int_{\Gamma} \frac{\partial \Phi}{\partial n_y}(x-y)w(y) ds_y, \quad x \in \Omega.$$

With V and W as above, if such functions v and w exist, they satisfy the first and second kind Fredholm integral equations [108]

$$Vv = g \quad \text{and} \quad \frac{1}{2}w + Ww = g \quad \text{on } \Gamma. \quad (2.53)$$

Writing $H^s = H^s(\Gamma)$, V and W are so-called pseudo differential operators of order -1 , i.e., bounded linear operators $H^s \rightarrow H^{s+1}$, in particular compact on H^s . The first kind equation is uniquely solvable provided a certain measure, the transfinite diameter δ_Γ of Γ , is such that $\delta_\Gamma \neq 1$, and the second kind equation in (2.53) always has a unique solution. Similar reformulations may be used also for Neumann boundary conditions, for a large number of other problems involving elliptic type equations, and for exterior problems; in fact, this approach to the numerical solution is particularly useful in the latter case.

In the Boundary Element Method (BEM) one determines the approximate solution in a piecewise polynomial finite element type space of a boundary integral formulation such as the above, using the Galerkin or the collocation method.

For the second kind equation in (2.53), using Galerkin's method and a finite dimensional subspace S_h of $L_2(\Gamma)$, we determine the discrete approximation $w_h \in S_h$ to w from

$$\frac{1}{2}\langle w_h, \chi \rangle + \langle Ww_h, \chi \rangle = \langle g, \chi \rangle, \quad \forall \chi \in S_h, \quad \text{where } \langle \cdot, \cdot \rangle = (\cdot, \cdot)_{L_2(\Gamma)}.$$

Writing $|\cdot|_S$ for the norm in $H^s(\Gamma)$, one has $|w_h - w|_0 \leq C_r(w)h^r$ if S_h is accurate of order $O(h^r)$, and by a duality argument one may show the superconvergent order negative norm estimate $|w_h - w|_{-r} \leq C_r(w)h^{2r}$; using an iteration argument this may be used to define an approximate solution \tilde{w}_h with $|\tilde{w}_h - w|_0 = O(h^{2r})$.

Consider for example the numerical solution of the first kind equation in (2.53) in the finite dimensional space S_h of periodic smoothest splines of order r , i.e., $S_h \subset C^{r-2}$ consists of piecewise polynomials in \prod_{r-1} . Our discrete problem is then to find $v_h \in S_h$ such that

$$\langle Vv_h, \chi \rangle = \langle g, \chi \rangle, \quad \forall \chi \in S_h.$$

It can be shown that [108] the bilinear form $\langle Vv, w \rangle$ associated with V is symmetric, bounded, and coercive with respect to the norm $|\cdot|_{-\frac{1}{2}}$ in a certain Sobolev space $H^{-\frac{1}{2}}(\Gamma)$, so that

$$\langle Vv, w \rangle = \langle v, Vw \rangle \leq C|v|_{-\frac{1}{2}}|w|_{-\frac{1}{2}} \quad \text{and} \quad \langle Vv, v \rangle \geq c|v|_{-\frac{1}{2}}^2, \quad \text{with } c > 0.$$

One may then show that [108]

$$|v_h - v|_{-\frac{1}{2}} \leq C \inf_{\chi \in S_h} |\chi - v|_{-\frac{1}{2}} \leq Ch^{r+\frac{1}{2}}|v|_r,$$

and a duality argument implies $|v_h - v|_{-r-1} \leq Ch^{2r+1}|v|_r$, where we use the norm in $H^{-r-1}(\Gamma)$. For x an interior point of Ω we therefore find for $u_h = Vv_h$ that $|u_h(x) - u(x)| \leq C_x|v_h - v|_{-r-1} \leq Ch^{2r+1}$, since $\Phi(x - y)$ is smooth when $y \neq x$.

Expressed in terms of a basis $\{\Phi_j\}$ of S_h this problem may be written in matrix form as $A\alpha = \tilde{g}$, where A is symmetric positive definite. However, although the dimension of A has been reduced by the reduction of the original two-dimensional problem to a one-dimensional one, in contrast to the finite element method for a differential equation problem, the matrix A is now not sparse. We also note that the elements $\langle V\Phi_i, \Phi_j \rangle$ require two integrations, one in forming $V\Phi_i$ and one in forming the inner product.

In order to reduce this work one may apply the collocation method and determine v_h from $Vv_h(x(s_j)) = g(x(s_j))$ at M_h quadrature points s_j in $[0, l]$, where $x = x(s)$ is a parametrization of Γ and $M_h = \dim(S_h)$.

In the vast literature on the numerical boundary integral methods much attention has been paid to the complications arising when our above regularity assumptions fail to be satisfied, such as for domains with corners in which case V and W are not compact.

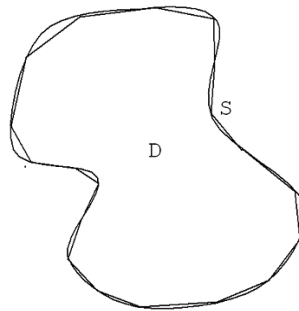


Figure-2.3 The boundary represented by a set of straight line panels.

There are a variety of techniques for deriving the system of linear equations from a given integral equation (see [109], [110], for example). In general, a method can be derived by replacing the integrals in an integral equation by a quadrature formula or by a weighted residual method such as the Galerkin method. Many methods for solving integral equation can be used to develop a particular boundary element method [111]. The application of collocation to a boundary integral equation requires that the boundary is represented by a set of panels. For example, a two-dimensional boundary can be approximated by a set of straight lines as illustrated in Figure-2.3. In order to complete the discretisation of the integral equations, the boundary functions also need to be approximated on each panel. In this work, it is the characteristics of the panel and the representation of the boundary function on the panel that together define the element in the boundary element method. Finally, by representing the boundary functions by a characteristic form on each panel, the boundary integral equations can be written as a linear system of equations.

CHAPTER - 3

PROJECT- 1

TWO-DIMENSIONAL FLUID FLOW PAST A RECTANGULAR PLATE

3.1 The Equations of Motion

The description of the flow of fluids is one of the richest and most challenging problems that can be treated on a computer. The non-linearity of the equations and the complexity of phenomena they describe (e.g., turbulence) sometimes make computational fluid dynamics more of an art than a science, and several book-length treatments are required to cover the field adequately. In this project, we will consider one relatively simple situation that can be treated by the relaxation methods for elliptic equations described in Chapter - 2 and that will serve to give some idea of the problems involved. This situation is the time-independent flow of a viscous, incompressible fluid past an object. For simplicity, we will take the object to be translationally invariant in one direction transverse to the flow, so that the fluid has a non-trivial motion only in two-coordinates, (x, y) ; this might describe a rod or beam placed in a steady flow of water. For the initial velocity we will consider two cases: the first is constant and the second is a varying function. Thus, in Case (1) we take the incident velocity equals V_0 and in Case (2) it is $V_0 e^{-x}$. We will also consider only the case where the cross-section of this rod is a rectangle with dimensions $2W$ transverse to the flow and T along the flow (see Fig. 3.1). This will greatly simplify the coding needed to treat the boundary conditions, while still allowing the physics to be apparent. We begin with an exposition of the basic equations and their discretization, follow with a brief discussion of the boundary conditions, and then give some guidance in writing the program and in extracting some understanding from it.

The non-linearity of the equations and the complexity of the flow of the fluid sometimes cause instability in the specific numerical method and divergence in the solution such as in turbulence phenomena [112]. The steady-state hydrodynamics in two dimensions, in our case, was considered and worked for different Reynolds-number. The Reynolds-number Re in our case of incompressible fluid, is physically defined as the ratio of mass diffusion time constant to mass convection time constant. This is mathematically defined as $Re = \frac{V_0 h}{\nu}$, where h is the step size dimension. It is seen that the Reynolds-number is inversely proportional to viscosity ν . The magnitude of the initial flowing velocity field was defined here by V_0 . Two fundamental equations are required for the mass density, ρ , and the velocity of the fluid element at each point in space, \mathbf{V} . The first of which is the continuity equation [113]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0, \quad (3.1)$$

and the second is the Navier-Stokes equation [113]

$$\frac{\partial \mathbf{V}}{\partial t} = -(\mathbf{V} \cdot \nabla) \mathbf{V} - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{V}, \quad (3.2)$$

where P is the pressure and ν is the kinematic viscosity, assumed constant. The first equation expresses the conservation of mass, and states that the density can change at a point in space only due to a net in-or out-flow of matter. The second equation expresses the conservation of momentum, and states that the velocity changes in response to convection, $(\mathbf{V} \cdot \nabla) \mathbf{V}$, spatial variations in the pressure, ∇P , and viscous forces $\nu \nabla^2 \mathbf{V}$. We will assume that the temperature is constant throughout the fluid.

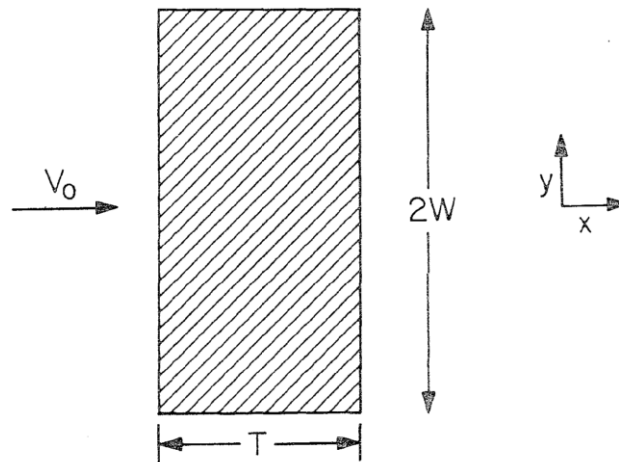


Figure-3.1 Geometry of the two-dimensional flow past a plate to be treated in this project.

Furthermore, we will be interested in studying time-independent incompressible fluid flows, so that equations (3.1) and (3.2) can be rewritten in the following forms

$$\nabla \cdot \mathbf{V} = 0, \quad (3.3)$$

$$(\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{V}. \quad (3.4)$$

For two-dimensional flow, these equations can be written explicitly in terms of the x and y components of the velocity field, denoted by u and v , respectively:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \quad (3.5)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \nabla^2 u; \quad (3.6)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \nabla^2 v. \quad (3.7)$$

Equations (3.5), (3.6) and (3.7) are three scalar equations for the fields u , v and P . While these equations could be solved directly, it is more convenient for two-dimensional problems to replace the velocity fields by two equivalent scalar fields: the stream function, $\psi(x, y)$, and the vorticity, $\zeta(x, y)$. The first of these is introduced as a convenient way of satisfying the continuity equation (3.5). The stream function is defined so that [100]

$$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x} \quad (3.8)$$

such a ψ function exists for all flows that satisfy the continuity equation. It can be seen also that \mathbf{V} is tangent to contour lines of constant ψ , the streamlines. The vorticity is given by

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad (3.9)$$

which is seen to be (the negative of) the curl of the velocity field. From the definition (3.8), it follows that ζ is related to the stream function ψ by

$$\nabla^2 \psi = \zeta \quad (3.10)$$

An equation for ζ can be derived by differentiating (3.6) with respect to y and (3.7) with respect to x , we get

$$\frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial y \partial x} + v \frac{\partial}{\partial y} (\nabla^2 u) \quad (3.11)$$

and

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + v \frac{\partial^2 v}{\partial y \partial x} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial x \partial y} + v \frac{\partial}{\partial x} (\nabla^2 v) \quad (3.12)$$

Subtracting (3.11) from (3.12) and invoking the continuity equation (3.5) and definition (3.8), we get

$$\begin{aligned} & \frac{\partial u}{\partial x} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + u \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \right) + \frac{\partial v}{\partial y} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + v \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} \right) \\ & = v \left(\frac{\partial}{\partial y} (\nabla^2 u) - \frac{\partial}{\partial x} (\nabla^2 v) \right) \\ & \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + u \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \right) + v \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} \right) \\ & = v \left(\frac{\partial}{\partial y} (\nabla^2 u) - \frac{\partial}{\partial x} (\nabla^2 v) \right) \\ & u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + v \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = v \left(\nabla^2 \frac{\partial u}{\partial y} - \nabla^2 \frac{\partial v}{\partial x} \right) = v \nabla^2 \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \end{aligned}$$

$$u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = v \nabla^2 \zeta.$$

Therefore

$$v \nabla^2 \zeta = \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y}. \quad (3.13)$$

Finally, an equation for the pressure can be derived by differentiating (3.6) with respect to x , and (3.7) with respect to y , we get

$$\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial x^2} + v \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.14)$$

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + v \frac{\partial^2 v}{\partial y^2} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial y^2} + v \frac{\partial}{\partial y} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3.15)$$

Now adding (3.14) to (3.15), we get

$$\begin{aligned} & \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + v \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 v}{\partial y^2} \\ & = -\frac{1}{\rho} \nabla^2 P + v \left(\frac{\partial}{\partial x} (\nabla^2 u) + \frac{\partial}{\partial y} (\nabla^2 v) \right) \\ & \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 - 2 \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial y} \left(\frac{\partial^2 \psi}{\partial x^2} \right) - \frac{\partial \psi}{\partial y} \frac{\partial^2}{\partial x^2} \frac{\partial \psi}{\partial y} + \left(\frac{\partial^2 \psi}{\partial y \partial x} \right)^2 \\ & \quad - \frac{\partial \psi}{\partial x} \left(\frac{\partial^2}{\partial x \partial y} \frac{\partial \psi}{\partial y} - \frac{\partial^2}{\partial y^2} \frac{\partial \psi}{\partial x} \right) = -\frac{1}{\rho} \nabla^2 P + v \nabla^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ & 2 \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 - 2 \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial y \partial x^2} - \frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial y \partial x^2} - \frac{\partial \psi}{\partial x} \left[\frac{\partial^2}{\partial x \partial y} \left(\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y} \right) \right] \\ & = -\frac{1}{\rho} \nabla^2 P \\ & \nabla^2 P = 2\rho \left\{ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right\} \end{aligned} \quad (3.16)$$

Equations (3.9), (3.13) and (3.16) are a set of non-linear elliptic equations, equivalent to the original equations (3.5), (3.6) and (3.7).

3.2 The Boundary Conditions

The boundary conditions on the centerline surfaces A and E of the plate are determined by symmetry. The y component of the velocity, v , must vanish on A and E, so that $\frac{\partial \psi}{\partial x}$ vanishes and accordingly A and E are streamlines. Moreover, since the normal velocity also vanishes on B, C, and D, the entire surface ABCDE is a single streamline. From symmetry, we can

also conclude that the vorticity vanishes on A and E. The upstream surface F is contiguous with the smoothly flowing incident fluid, so that we can put

$$v = -\frac{\partial\psi}{\partial x} = 0; \quad \zeta = 0 \quad \text{on F.} \quad (3.17)$$

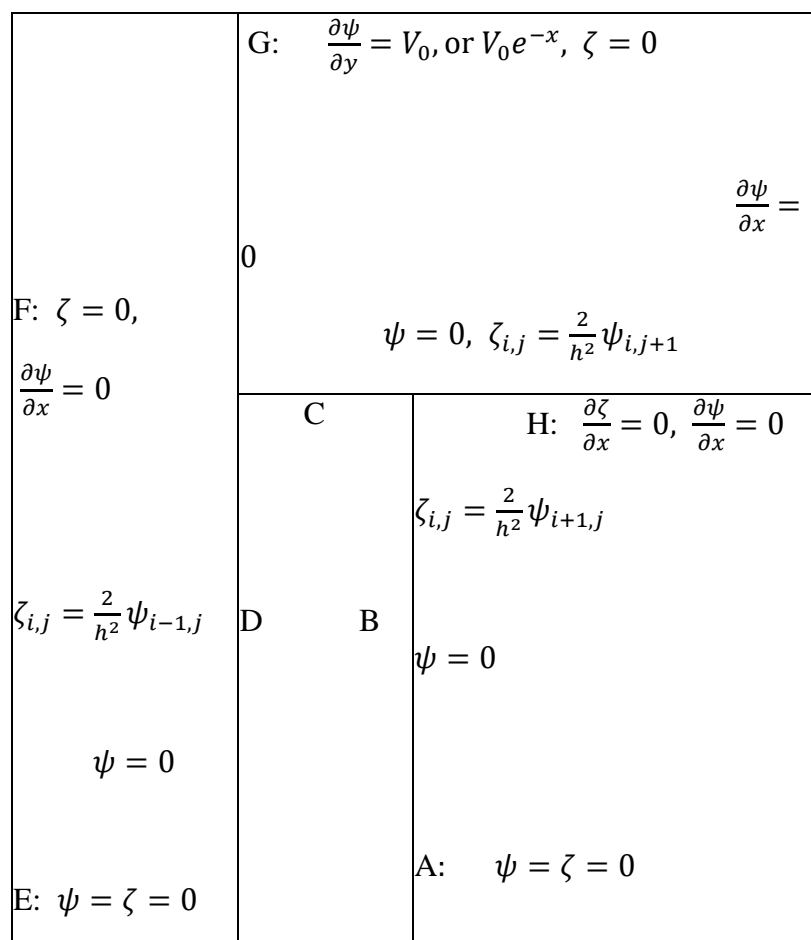


Figure-3.2 Boundary conditions on ψ and ζ for the flowing system and the obstacle for the upper part, in symmetry.

The boundary conditions on the upper boundary G are similarly straightforward. We may expect G to be in free flow, if the lattice is large enough. Hence,

$$u = \frac{\partial\psi}{\partial y} = V_0 \text{ or } V_0e^{-x}; \quad \zeta = 0 \quad \text{on } G, \quad (3.18)$$

is one appropriate choice.

The downstream boundary H is much more ambiguous and, so long as it is sufficiently far from the plate one convenient choice is

$$\frac{\partial \psi}{\partial x} = \frac{\partial \zeta}{\partial x} = 0 \quad \text{on } H. \quad (3.19)$$

At the walls of the plates, (B , C , and D) one of the correct boundary conditions is that

$$v = -\frac{\partial \psi}{\partial x} = 0 \quad \text{on } B, C, D. \quad (3.20)$$

However, the other boundary condition appropriate for viscous flow is that the tangential velocity be zero,

$$u = \frac{\partial \psi}{\partial y} = 0 \quad \text{on } B, C, D. \quad (3.21)$$

The above boundary conditions are illustrated in Figure-3.2.

The boundary conditions for the pressure on all surfaces are of the Neumann type, and follow from equations (3.6) and (3.7). From the symmetry $\frac{\partial P}{\partial y} = 0$ on the centerlines A and E.

3.3 The Method of Solution

To solve equations (3.9), (3.13) and (3.16) numerically we introduce a two-dimensional lattice of uniform spacing h having N_x and N_y points in the x and y directions, respectively, and use the indices i and j to measure these coordinates (see Figure-3.3). It is convenient to scale the equations by measuring all lengths in units of h and all velocities in units of the magnitude of the incident fluid velocity, V_0 . The stream function is then measured in units of $V_0 h$, while the vorticity is in units of $\frac{V_0}{h}$, and the pressure is conveniently scaled by ρV_0^2 . The second step is to differencing equations (3.9), (3.13) and (3.16) by using symmetric second- and first-difference operators [114]. Accordingly, the lattice Reynolds number, $Re = \frac{V_0 h}{\nu}$, is a dimensionless measure of the strength of the viscous forces.

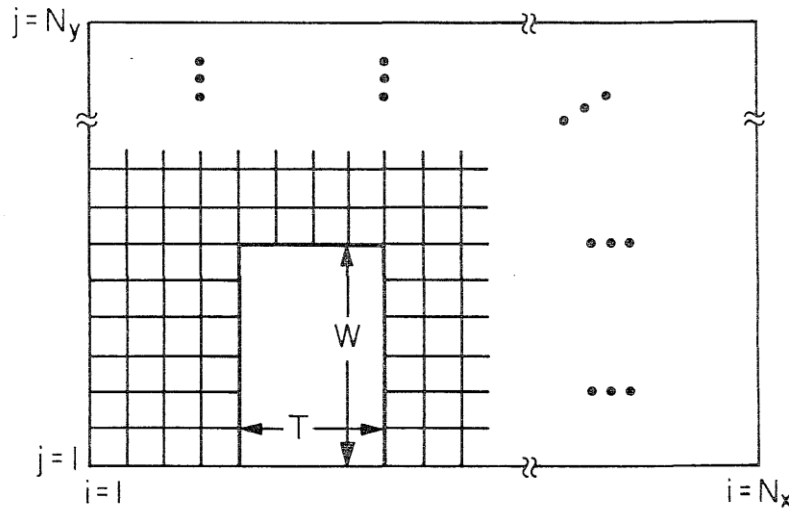


Figure-3.3 The lattice to be used in calculating the fluid flow past the plate illustrated in Figure-3.1.

Our numerical method of solution of the resulting coupled non-linear elliptic partial differential equations for the stream function and the vorticity is by using the relaxation method iteratively [6]. In the model, a rectangular obstacle was defined with height $16h$ and width $8h$. Since, for simplicity, the incident fluid was in free-flowing case parallel and far enough from the obstacle, the symmetry property was used. For solving the upper part of the flowing system in lattice space we take $N_x = 70$, $N_y = 24$, the vorticity relaxation = 0.3, and the stream relaxation = 0.3. Furthermore, the lattice Reynolds number was allowed to take on the values from 0.5 to 300. We begin the iteration scheme by choosing trial values corresponding to the free-flowing solution $\psi = y$ and $\zeta = 0$. We then perform one relaxation sweep of the first equation to get an improved value of ψ .

Using the finite differencing scheme, the first equation of (3.9) can be written as follows

$$\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j} = h^2 \zeta_{i,j}. \tag{3.22}$$

The next step is to transform the equations to dimensionless ones in ψ and ζ by introducing new variables ψ' and ζ' , defined by $\psi = \psi' V_0 h$ and $\zeta = \frac{V_0}{h} \zeta'$, so that [115]

$$\psi'_{i+1,j} + \psi'_{i-1,j} + \psi'_{i,j+1} + \psi'_{i,j-1} - 4\psi'_{i,j} = \frac{h}{V_0} \zeta_{i,j} = \zeta'_{i,j}. \tag{3.23}$$

Therefore, the values of ψ' and ζ' are in units of $V_0 h$ and $\frac{V_0}{h}$.

The right-hand side of equation (3.13) can be similarly written as

$$\zeta'_{i+1,j} + \zeta'_{i-1,j} + \zeta'_{i,j+1} + \zeta'_{i,j-1} - 4\zeta'_{i,j} - \frac{Re}{4} \left[(\zeta'_{i+1,j} - \zeta'_{i-1,j})(\psi'_{i,j+1} - \psi'_{i,j-1}) - (\zeta'_{i,j+1} - \zeta'_{i,j-1})(\psi'_{i+1,j} - \psi'_{i-1,j}) \right] = 0. \tag{3.24}$$

The two coupled partial differential equations (3.23) and (3.24) can be written in the following form [115]

$$\begin{aligned}
 (a1)_{i+1,j}\psi'_{i+1,j} + (b1)_{i-1,j}\psi'_{i-1,j} + (c1)_{i,j+1}\psi'_{i,j+1} \\
 + (d1)_{i,j-1}\psi'_{i,j-1} + (e1)_{i,j}\psi'_{i,j} &= (f1)_{i,j}, \\
 (a2)_{i+1,j}\zeta'_{i+1,j} + (b2)_{i-1,j}\zeta'_{i-1,j} + (c2)_{i,j+1}\zeta'_{i,j+1} \\
 + (d2)_{i,j-1}\zeta'_{i,j-1} + (e2)_{i,j}\zeta'_{i,j} &= (f2)_{i,j}
 \end{aligned} \tag{3.25}$$

where the corresponding coefficient-matrix elements are defined by [115]:

$$\begin{aligned}
 a1_{i+1,j} &= 1, & a2_{i+1,j} &= 1 - \frac{Re}{4} [\psi'_{i,j+1} - \psi'_{i,j-1}], \\
 b1_{i-1,j} &= 1, & b2_{i-1,j} &= 1 + \frac{Re}{4} [\psi'_{i,j+1} - \psi'_{i,j-1}] \\
 c1_{i,j+1} &= 1, & c2_{i,j+1} &= 1 - \frac{Re}{4} [\psi'_{i+1,j} - \psi'_{i-1,j}], \\
 d1_{i,j-1} &= 1, & d2_{i,j-1} &= 1 - \frac{Re}{4} [\psi'_{i+1,j} - \psi'_{i-1,j}] \\
 e1_{i,j} &= -4, & e2_{i,j} &= -4, & f1_{i,j} &= \zeta'_{i,j}, & f2_{i,j} &= 0
 \end{aligned} \tag{3.26}$$

These constant coefficients are such that the arrays declaration were avoided and simply replaced by their corresponding constant values. The coefficients matrix elements have the same dimensions.

Furthermore, the velocity field $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$ was related to the stream function $\psi(x, y)$, but by using central differencing scheme for the equations

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\psi'_{v_0h}}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x} = -\frac{\partial\psi'_{v_0h}}{\partial x}. \tag{3.27}$$

Accordingly

$$u_{i,j} = \frac{1}{2}(\psi'_{i,j+1} - \psi'_{i,j-1}), \quad v_{i,j} = -\frac{1}{2}(\psi'_{i+1,j} - \psi'_{i-1,j}). \tag{3.28}$$

The boundary conditions are, now, simply given by

$$\text{At boundary A, } \psi' = 0, \quad \psi'_{i,j} = 0, \quad \zeta' = 0, \quad \zeta'_{i,j} = 0$$

$$\text{At boundary B, } \psi'_{i,j} = 0, \quad \zeta'_{i,j} = 2\psi'_{i+1,j}$$

$$\text{At boundary C, } \psi'_{i,j} = 0, \quad \zeta'_{i,j} = 2\psi'_{i,j+1}$$

$$\text{At boundary D, } \psi'_{i,j} = 0, \quad \zeta'_{i,j} = 2\psi'_{i-1,j}$$

At boundary E, $\psi' = 0$ $\psi'_{i,j} = 0$ $\zeta' = 0$ $\zeta'_{i,j} = 0$

At boundary F, because of the initial parallel free flow assumption and using forward finite differencing scheme, $\frac{\partial \psi}{\partial x} = 0$ $\psi'_{i-1,j} = \psi'_{i,j}$ $\zeta' = 0$ $\zeta'_{i,j} = 0$

At boundary G, because of the far away from the obstacle and using backward finite differencing scheme

$$\frac{\partial \psi}{\partial y} = V_0, \text{ or } V_0 e^{-x}, \quad \psi'_{i,j} - \psi'_{i,j-1} = 1, \quad \zeta' = 0, \text{ and } \zeta'_{i,j} = 0$$

At boundary H, because of the far away enough from the obstacle, the boundary conditions were assumed that the parallel free flow was restored so that, by using backward finite differencing scheme, we get

$$\text{Set-1 condition: } \frac{\partial \psi}{\partial x} = 0 \quad \psi'_{i,j} = \psi'_{i-1,j} \quad \frac{\partial \zeta}{\partial x} = 0 \quad \zeta'_{i,j} = \zeta'_{i-1,j}$$

$$\text{Set-2 condition: } \frac{\partial \psi}{\partial x} = 0 \quad \psi'_{i,j} = \psi'_{i-1,j} \quad \zeta = 0 \quad \zeta'_{i,j} = 0$$

In the program both of the two sets of boundary conditions at H were used and the results were reported.

Accordingly, the successive over relaxation (SOR) method [6] was used. It is of interest to notice that for $10 < i < 18$ and $0 < j < 8$, there was forbidden region. For the SOR method, the residue vectors have the following magnitudes

$$RES(\psi)_{i,j} = \psi'_{i,j} - \frac{\omega(\zeta_{i,j})_{\psi}}{(e1)_{i,j}} \quad RES(\zeta)_{i,j} = \zeta'_{i,j} - \frac{\omega(\zeta_{i,j})_{\zeta}}{(e2)_{i,j}},$$

where ω is the over-relaxation factor [6]. The steady state of the relaxation was approached and $RES(\psi)$ were lesser than 10^{-6} . The algorithm used in the program was firstly to initialize the stream function ψ and the vorticity ζ . The pressure P was initialized as for parallel free of flow over the space excluding the forbidden region.

$$\psi'_{i,j} = j \quad ; \quad \zeta'_{i,j} = 0$$

Then, the coefficient matrices of the partial differential equations for the stream function were updated and followed by doing the relaxation for the stream function at the interior points. After the relaxation has been made once for the stream function, the boundary conditions must be checked and then the coefficient matrices of the partial differential equations are to be calculated at the interior points. The procedures were iteratively repeated until both magnitudes of the residue vectors: $Res(\zeta)$ and $Res(\psi)$ converged and have values less than 10^{-6} . The over-relaxation factor ω was used and assumed the value of 0.7 which was out of the range $1 \leq \omega \leq 2$. This is because the nonlinear set of partial differential equations for the vorticity function is sensitively diverge for the over-relaxation. For our case, the larger values of the Reynolds-number are used and the smaller value of the allowed maximum over-relaxation factor ω was found. Therefore, for the

comparison of the convergence rate for different situations, the values of ω were 0.7 and 0.2, for $Re = 300$.

3.4 Results and Conclusions

A computer program is written in FORTRAN to solve for ψ, ζ , and P subject to the boundary conditions discussed in section-3.2 and the numerical method of solution discussed in section-3.3 of this chapter. In Case (1), we present the resulting values when the initial velocity is constant and equals V_0 . In Case (2), we present the resulting values when the initial velocity is variable and equals $V_0 e^{-x}$.

Case (1): The initial velocity is constant and equals V_0

In Table-3.1 we present the resulting values of the pressure force, the viscous force, and the minimum and maximum values of the vorticity and the stream function. The iteration step is given in the first column of Table-3.1.

It is seen from the resulting values in the last two rows that our method converges and gives exactly the same values after the 5000-iteration step.

It is of interest to notice that the pressure force is equal to $(0.6318 \rho V_0^2)$, while the viscous force is equal to $(0.3323 Re)$. The vorticity varies between $(-0.0145 \frac{V_0}{h})$ and $(0.5109 \frac{V_0}{h})$.

Table-3.1 The resulting values of the pressure force, the viscous force, and the minimum and maximum values of the vorticity and the stream function. The iteration step is given in the first column.

N Iterations	Pressure Force	Viscous Force	Vorticity		Stream Function	
			min.	max.	min.	max.
1	75.0459	8.5549	0.0000	11.1900	0.0000	19.000
10	17.0205	4.1333	0.0000	5.6470	0.0000	19.000
20	4.8771	2.3708	0.0000	4.2160	0.0000	19.000
30	2.7476	1.7373	-0.2171	3.5060	-0.1085	19.000
40	2.2888	1.4435	-0.2798	3.0690	-0.1399	19.000
50	2.1410	1.2765	-0.2813	2.7620	-0.2007	19.000
60	2.0657	1.1681	-0.2667	2.5300	-0.2326	19.000
70	2.0142	1.0918	-0.2475	2.3460	-0.2392	19.000
80	1.9736	1.0349	-0.2272	2.1970	-0.2673	18.990
90	1.9387	0.9907	-0.2080	2.0730	-0.2771	18.980
100	1.9071	0.9551	-0.1906	1.9690	-0.2766	18.970
200	1.6509	0.7508	-0.1018	1.4330	-0.3063	18.550
300	1.4648	0.6879	-0.0670	1.2170	-0.1732	17.850
400	1.3158	0.6272	-0.0507	1.0840	-0.0968	17.100
500	1.1918	0.5784	-0.0427	0.9842	-0.0700	16.400

600	1.0885	0.5372	-0.0381	0.9018	-0.0574	15.760
700	1.0033	0.5023	-0.0345	0.8323	-0.0484	15.180
800	0.9335	0.4728	-0.0314	0.7735	-0.0412	14.680
900	0.8766	0.4481	-0.0288	0.7240	-0.0359	14.250
1000	0.8303	0.4275	-0.0265	0.6824	-0.0312	13.870
2000	0.6555	0.3445	-0.0162	0.5322	-0.0124	12.200
3000	0.6345	0.3337	-0.0147	0.5134	-0.0101	11.950
4000	0.6321	0.3325	-0.0148	0.5111	-0.0098	11.920
5000	0.6318	0.3323	-0.0147	0.5109	-0.0098	11.920
6000	0.6318	0.3323	-0.0145	0.5109	-0.0098	11.920

The number of iterations, N , required for the convergence of the solutions for the different Reynolds-numbers, Re , together with the over-relaxation factor ω , was shown in Table-3.2, together with the maximum and minimum values of the computed functions ψ , ζ , P , and the velocity magnitude V .

Table-3.2 The number of iterations, N , required for convergence of solutions for different Reynolds-numbers Re , together with the over-relaxation factor ω . The maximum and minimum values of the computed functions are also given.

Re	ω	N	Stream function ψ		Vorticity function ζ		Static Pressure P		Velocity magnitude	
			Max	Min	Max	Min	Max	Min	Max	Min
0.5	0.7	2330	0.003	0	0.003297	1×10^{-7}	7.1×10^{-6}	-2×10^{-6}	27.8×10^{-6}	0
1.0	0.7	536	0.006	0	0.00728	3×10^{-7}	146×10^{-4}	-6×10^{-7}	555×10^{-4}	0
10	0.7	458	0.0614	0	0.1558	53×10^{-7}	2.49×10^{-4}	-168×10^{-6}	555×10^{-6}	0
20	0.7	374	0.1242	0	0.3882	1×10^{-5}	7.5×10^{-4}	-7.9×10^{-6}	116×10^{-5}	0
100	0.7	451	0.679	0	2.327	2×10^{-5}	0.0158	-0.0165	597×10^{-5}	0
200	0.7	3120	1.432	0	4.677	7×10^{-5}	0.065	-0.052	0.012	0
300	0.2	893	2.194	0	7.05	26×10^{-5}	0.1466	-0.1066	0.018	0

In Figures-3.4 to 3-18, we present the plots of the static pressure, the vorticity magnitude and the stream function for Reynolds number $Re = 10, 20, 100, 200,$ and 300 , respectively.

From the results of Table-3.2 we notice that for large values of the Reynolds number, the pressure in front of the obstacle increases knowing that when velocity increases the pressure is decreasing so that the pressure above the obstacle decreases when the velocity decreases. The velocity magnitude increased with increasing the Reynolds number. Also, at high Reynolds numbers the position of high velocity of fluid are forward with the flow.

The number of iterations used to approach the steady state of the relaxation was found that, for larger value of Reynolds number, the number of iterations for convergence of the stream function was slower than that for the vorticity function. The number of iterations used for vorticity function indicates that the first consumed iterations for the vorticity function reached and stayed within the convergence limit. The vorticity function was still refreshed differently for different iteration step because the dependence of the stream function was slower than that of the vorticity function. The results of this case are published in [115]

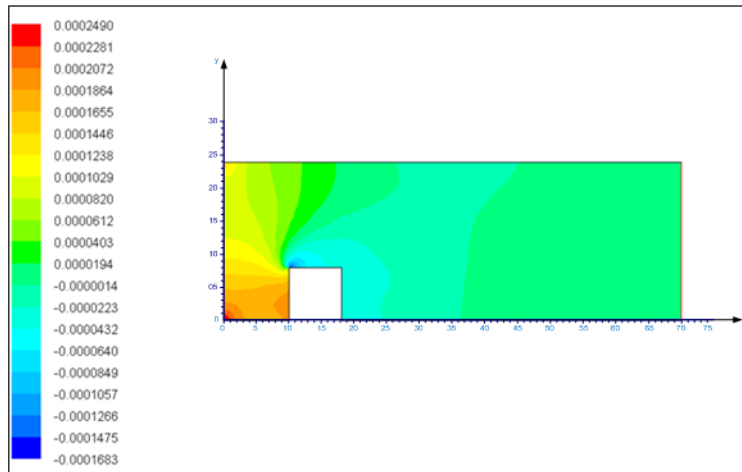


Figure-3.4 Counters of static pressure for Reynolds number $Re = 10$

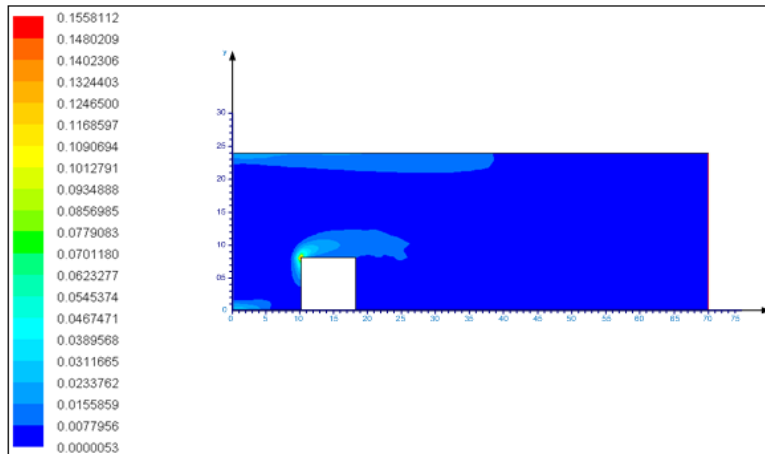


Figure-3.5 Counters of vorticity magnitude (1/s) for Reynolds number $Re = 10$

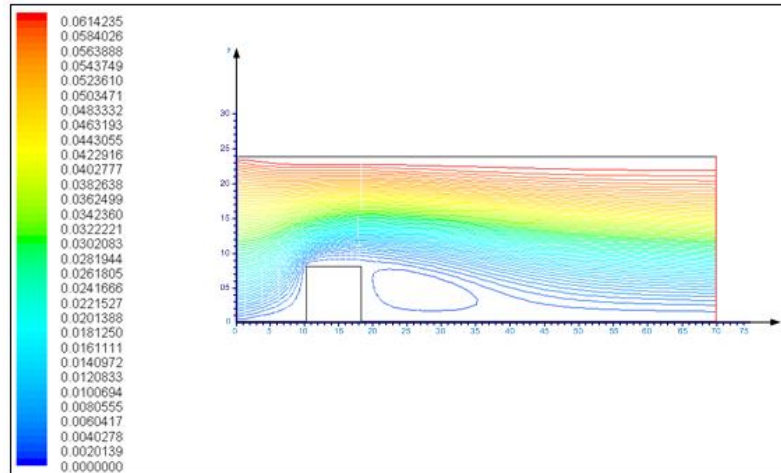


Figure-3.6 Stream function (Kg/s) for Reynolds number $Re = 10$

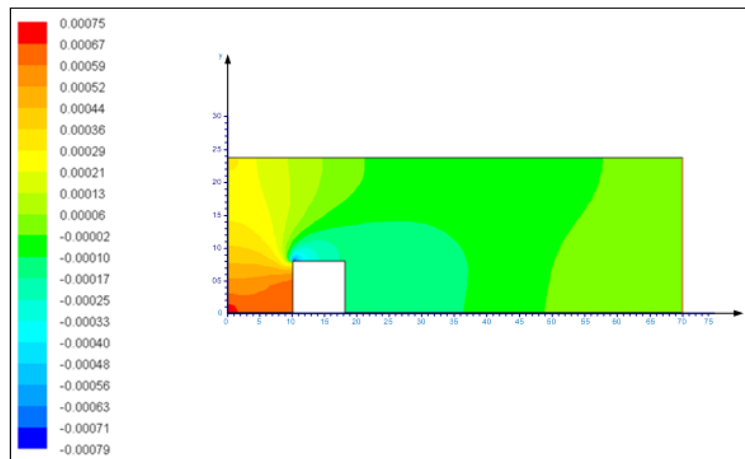


Figure-3.7 Counters of static pressure for Reynolds number $Re = 20$

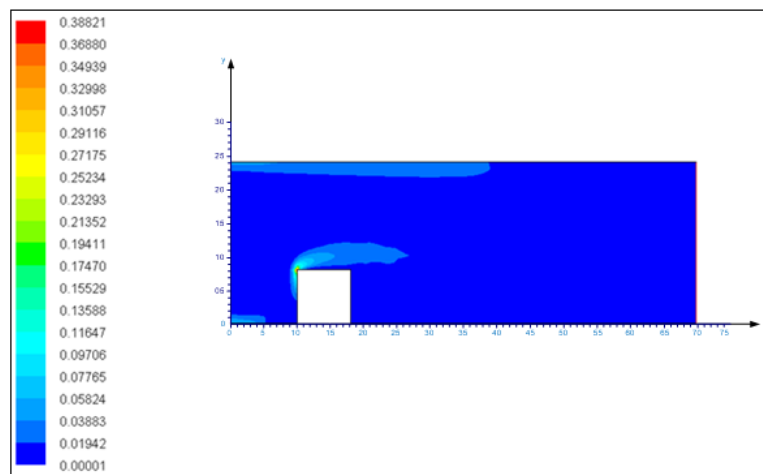


Figure-3.8 Counters of vorticity magnitude (1/s) for $Re = 20$

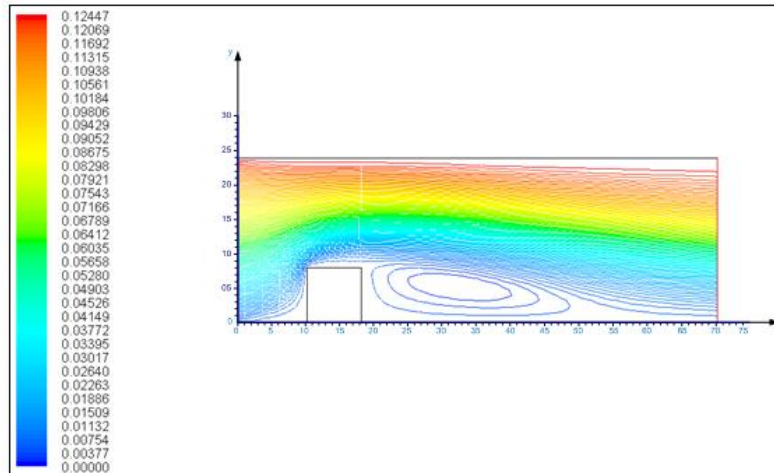


Figure-3.9 Stream function (Kg/s) for Reynolds number $Re = 20$

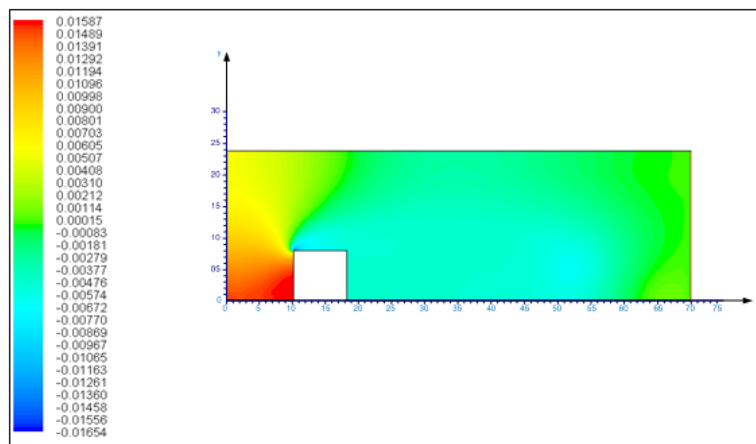


Figure-3.10 Counters of static pressure for Reynolds number $Re = 100$

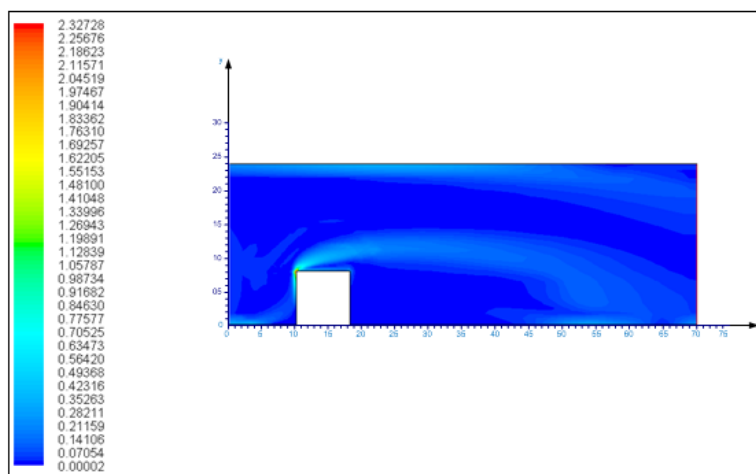


Figure-3.11 Counters of vorticity magnitude (1/s) for Reynolds number $Re = 100$

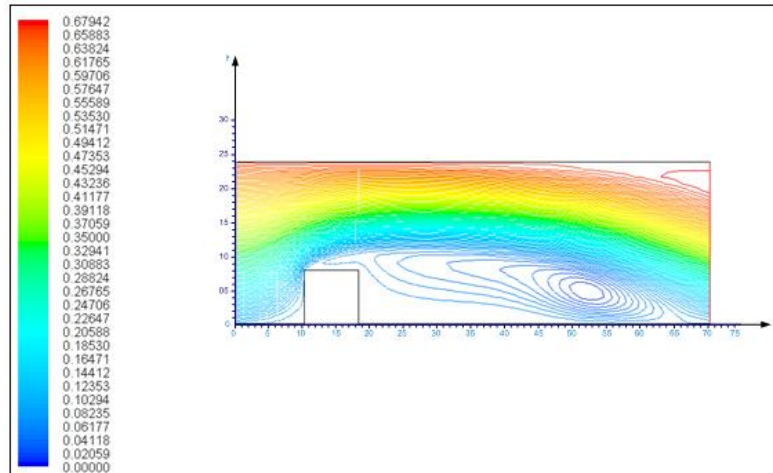


Figure-3.12 Stream function (Kg/s) for Reynolds number $Re = 100$

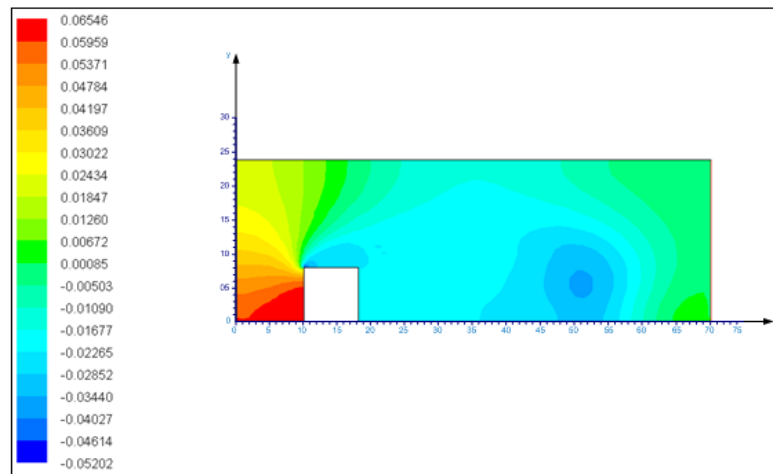


Figure-3.13 Counters of static pressure for Reynolds number $Re = 200$

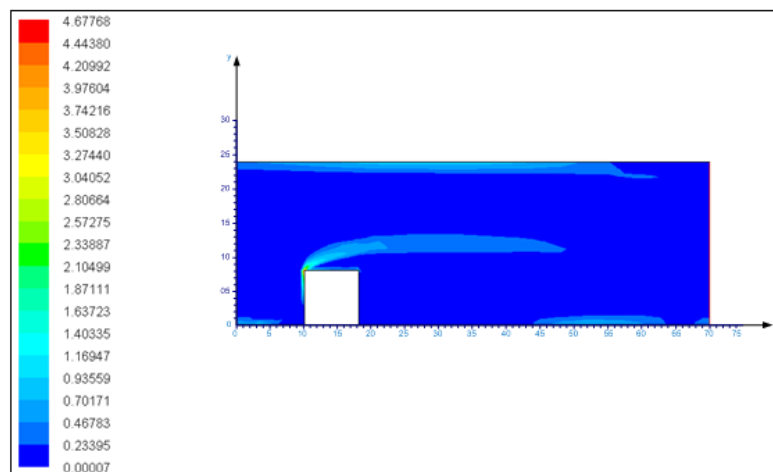


Figure-3.14 Counters of vorticity magnitude (1/s) for Reynolds number $Re = 200$

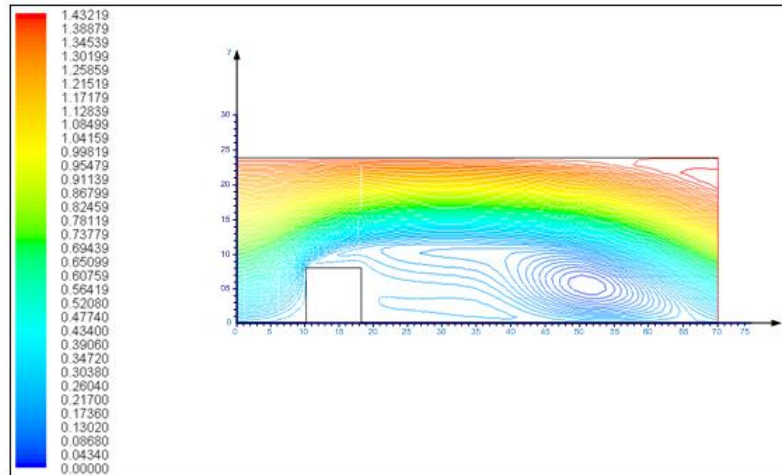


Figure-3.15 Stream function (Kg/s) for Reynolds number $Re = 200$

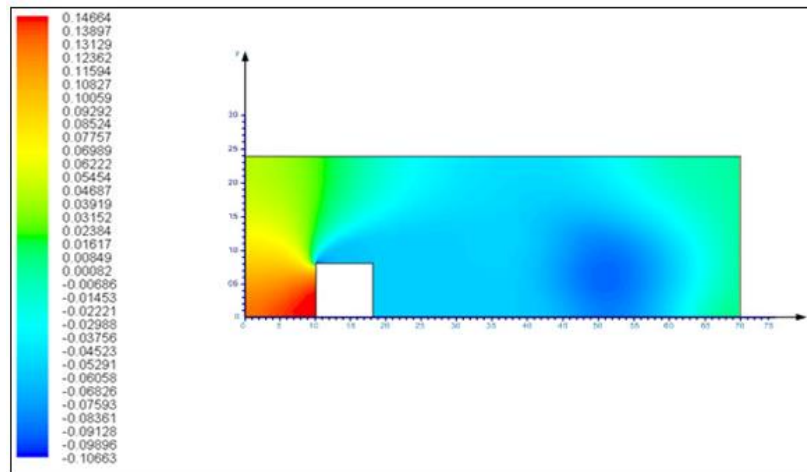


Figure-3.16 Counters of static pressure for Reynolds number $Re = 300$

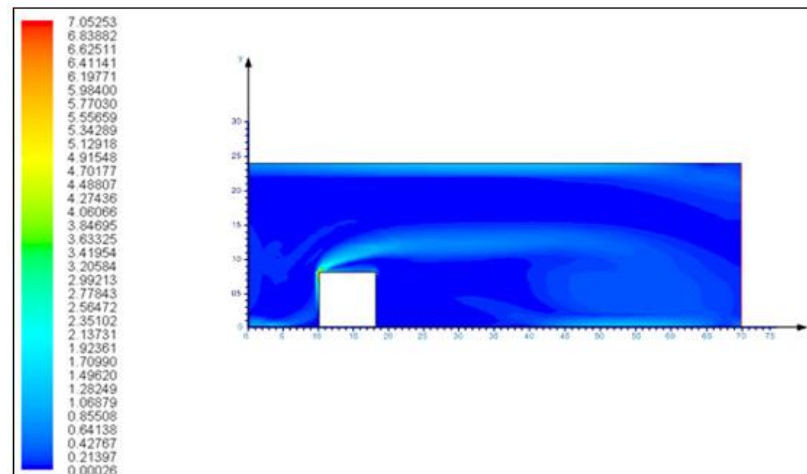


Figure-3.17 Counters of vorticity magnitude (1/s) for Reynolds number $Re = 300$

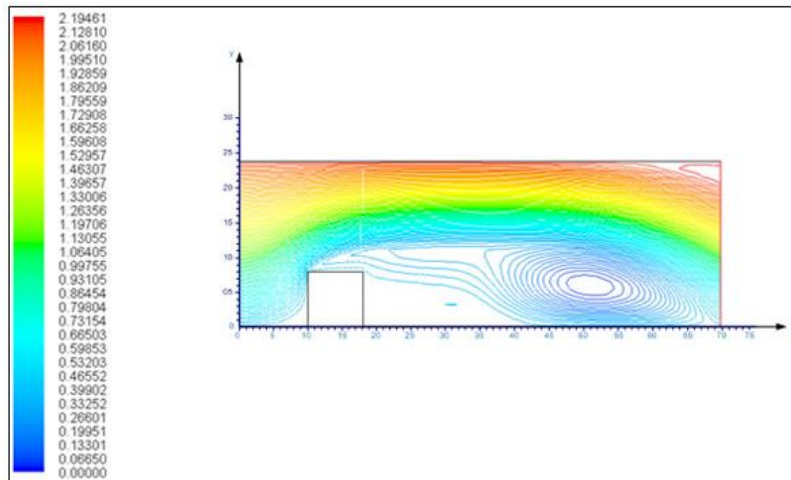


Figure-3.18 Stream function (Kg/s) for Reynolds number $Re = 300$

Case (2): The initial velocity is variable and equals $V_0 e^{-x}$

The number of iterations, N , required for the convergence of the solutions for the different Reynolds-numbers, Re , together with the over-relaxation factor ω , was shown in Table-3.3, together with the maximum and minimum values of the computed functions ψ , ζ , P , and the velocity magnitude V .

Table-3.3 the number of iterations used for convergence of the solutions and their related minimum and maximum values for $v = V_0 e^{-x}$, decaying flow in the $x -$ direction.

Re	ω	N	Stream function ψ		Vorticity magnitude ζ		Static Pressure P		Velocity magnitude	
			Max	Min	Max	Min	Max	Min	Max	Min
0.5	0.7	390	0.003	0	0.0025	1×10^{-7}	7.3×10^{-6}	-2×10^{-7}	27.7×10^{-6}	0
1.0	0.7	390	0.006	0	0.0056	0	15×10^{-6}	-1×10^{-6}	555×10^{-4}	0
10	0.7	290	0.0613	0	0.1187	5×10^{-6}	2.55×10^{-4}	-135×10^{-6}	5.52×10^{-4}	0
20	0.7	260	0.1242	0	0.294	4×10^{-5}	7.6×10^{-4}	-6.4×10^{-4}	0.00114	0
100	0.7	480	0.669	0	2.017	5.8×10^{-5}	0.0156	$-14.9 \text{ e-}3$	0.00597	0
200	0.7	340	1.404	0	4.205	2.8×10^{-5}	0.054	$-5.1 \text{ e-}2$	0.0119	0
300	0.2	9450	1.955	0	6.415	18.7×10^{-5}	0.1912	-0.062	0.0178	0

In Figures-3.19 to 3.46 we present the variations of the static pressure, the velocity magnitude, the vorticity magnitude and the stream function, as functions of position, for $Re = 0.5, 1, 10, 20, 100, 200$ and 300 , respectively [115].

From the results we notice that for large values of the Reynolds number, the pressure in front of the obstacle increases, knowing that when velocity increases the pressure is decreasing so that the pressure above the obstacle decreases when the velocity decreases.

The velocity magnitude will be increased with increasing the Reynolds number. Also, at high Reynolds numbers the position of high velocity of fluid is translated forward with the flow.

The convergence rates of the pressure function for different Reynolds number is seen in the figures.

The number of iterations used to approach the steady state of the relaxation shows that, for larger value of the Reynolds number, the number of iterations for convergence of the stream function was slower than that for the vorticity function. The number of iterations for the vorticity function, indicates that the first consumed iterations for the vorticity function is reached and stayed within the convergence limit. The vorticity function was still refreshed differently for different iteration step because the dependence of the stream function was slower than that of the vorticity function. The convergence rates of the stream function and the vorticity function for different Reynolds number is shown in the figures.

The convergence rates in case - 1 [114], with initial velocity equals V_0 , were found to be closed to that of our present situation, but the overall number of iterations is larger than that of the previous case in [114]. In general, for more strictly conditions such as in the present case, the used iterations could be larger comparing to the previous case of [114]. Also, the same effects are obtained for greater value of Reynolds number.

The obtained results for Reynolds number $Re = 1.0$ in the present case, show that the number of iterations was, abnormally, lesser than that of the previous case [114]. From Figure-3.25, we notice, for the steady-state flowing system of Reynolds number $Re = 1.0$ of vorticity near the boundary, that the tail of the vorticity near the boundary H was distorted due to the new condition at G. This implied that the dimension in boundary A, was not sufficiently large enough. Therefore, by using the new condition at higher value of the Reynolds number, the dimensions of the artificial boundaries required to be larger and involved more computational work.

The low Reynolds number flowing system, for high value of viscosity, was actually a lubricant flow. For the Reynolds number $Re = 0.5$, according to Figure-3.22 for the stream function contour, the flowing streamlines were smoothly overcome the obstacle and the shape was likely symmetric about the mid-axis but shifted to the right of the obstacle along the y-direction. Considering the plot of the vorticity, Figure-3.21, the local maximum vortex were observed near the point-edge ($i = 10, j = 8$) and ($i = 18, j = 8$), and the rotational axes were found to be pointing out from the paper of the graph (z-direction). There was no curl enter (eddy current) observed behind the obstacle for Reynolds's number $Re = 0.5$ situation.

Comparing the vorticity contour plots for Reynolds number $Re = 0.5, 1, \text{ and } 10$, Figures-3.21, 3.25 and 3.29, respectively, the front section was found to be compressed and the tail was elongated as the Reynolds number increased. For the higher Reynolds

number system, the greater distortion was observed. For the Reynolds number $Re = 0.5$, the positive maximum value of the vertex at the point edge was shifted to the front one and the latter one disappeared or is difficult to be observed.

However, there was a negative vortex, for Reynolds number $Re = 0.5, 1.0$ and 10 . The negative value of the vortex means that the curl exists but the rotation axis is anti-parallel to the z -direction (pointed into the paper) and the curl was eddy current.

According to the results of the velocity curves, Figures-3.20, 3.24 and 3.28, for Reynolds number $Re = 0.5, 1.0$ and 10 , there were found center of the curl and are found to be behind to the obstacle.

The center of the eddy was found to be shifted to the right for larger Reynolds number, which was different from the vortex near the point edge of the obstacle. The eddy was found to be larger value (more negative) for larger Reynolds number.

For the classical fluid model, when there is a low-density region created behind the obstacle, the fluid in the nearby region would be similar to the fluid into this low density region. The fluid flow into this region and the pattern was different for different Reynolds number systems. For low Reynolds number, it has large viscosity. This is possible for the lubricant flow, which is massive, such as for polymers or highly intermolecular force. The neighborhoods are highly effective to the flowing path, therefore the laminar flow is observed, for the high Reynolds number system (the larger value of the ratio of mass-diffusion time constant to mass-convection time constant). This implies that the jet existing through the divergent side and recirculation zones are easily appeared and the reversibility for the Reynolds number system is reduced. Therefore, for higher Reynolds number, the steady state solution is more difficultly to be approached due to the lowered reversibility.

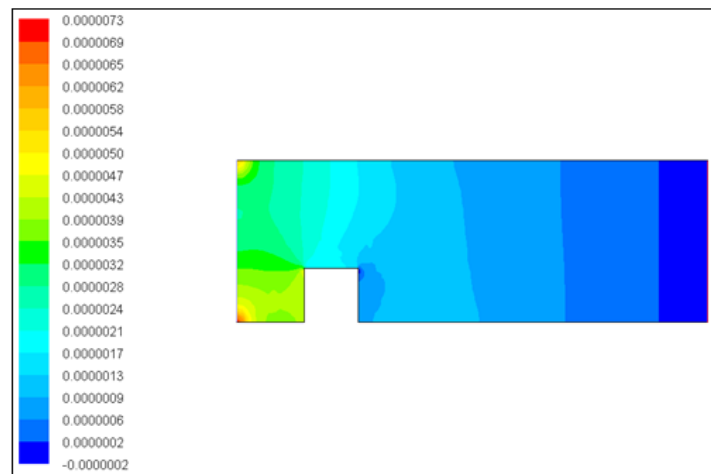
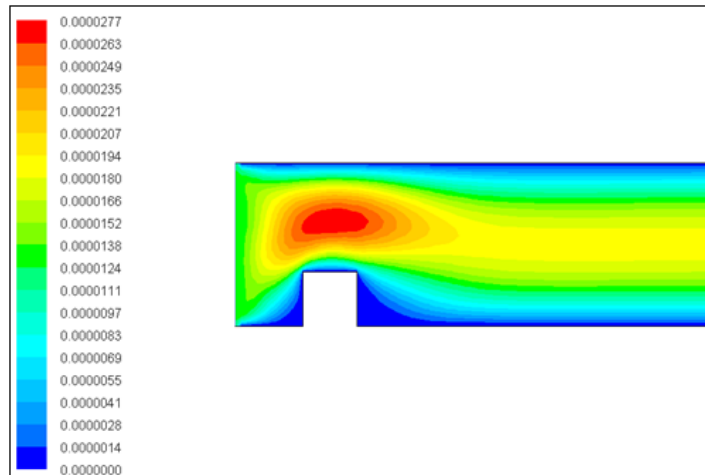
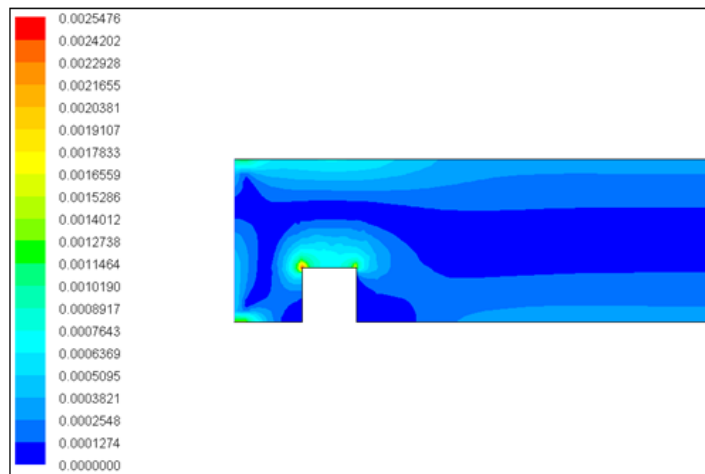
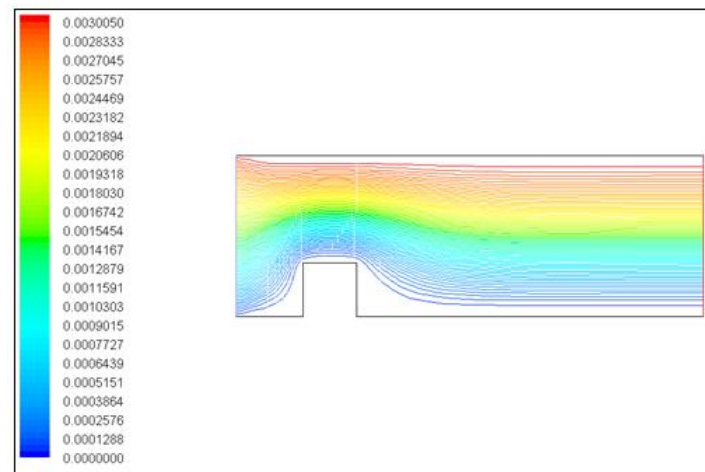
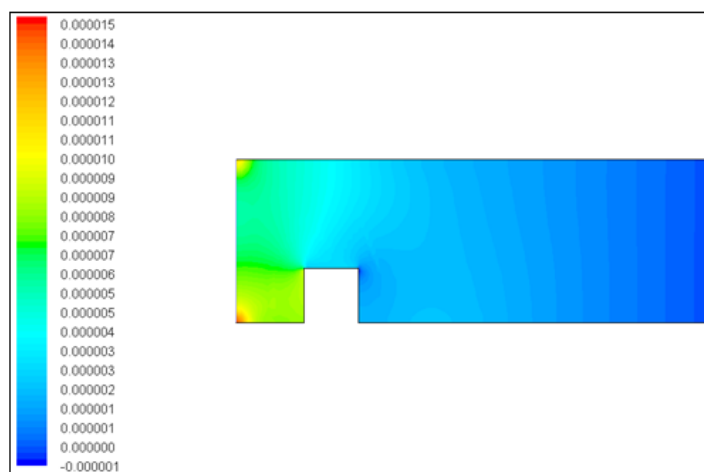
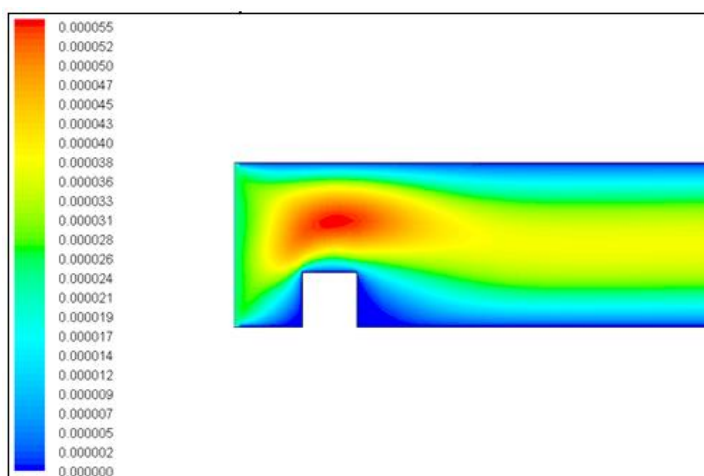
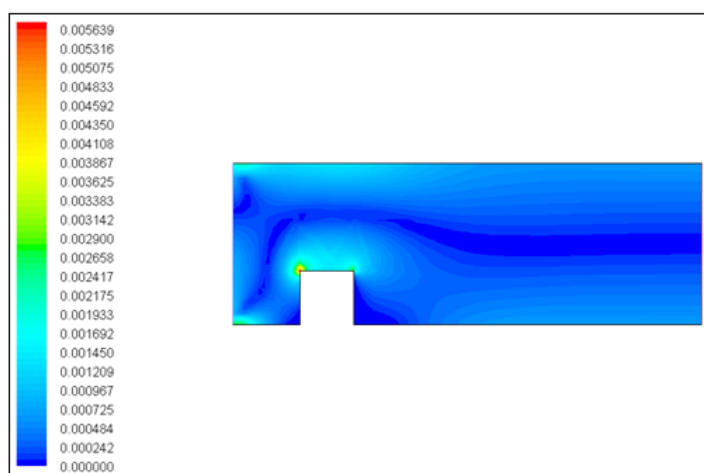


Figure-3.19 Static pressure ($Re = 0.5$)

Figure-3.20 Velocity magnitude ($Re = 0.5$)Figure-3.21 Vorticity magnitude (1/s) ($Re = 0.5$)Figure-3.22 Stream function (Kg/s) ($Re = 0.5$)

Figure-3.23 Static pressure ($Re = 1$)Figure-3.24 Velocity ($Re = 1$)Figure-3.25 Vorticity magnitude (1/s) ($Re = 1$)

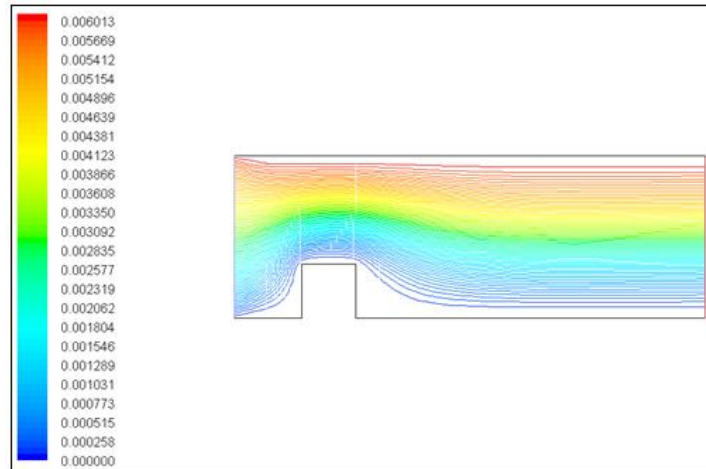


Figure-3.26 Stream function (Kg/s) ($Re = 1$)

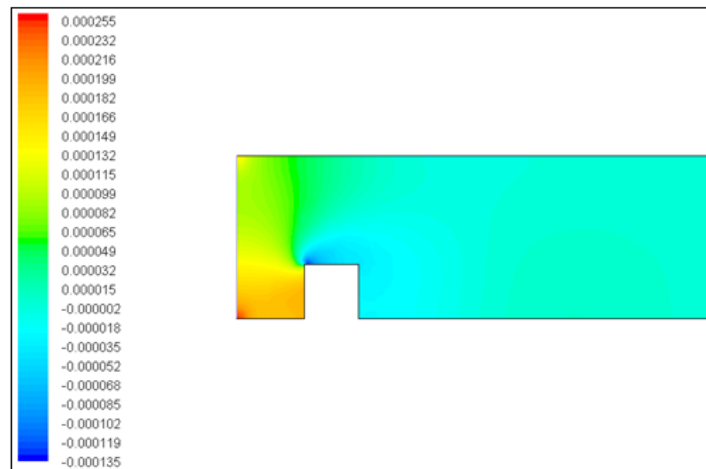


Figure-3.27 Static pressure ($Re = 10$)

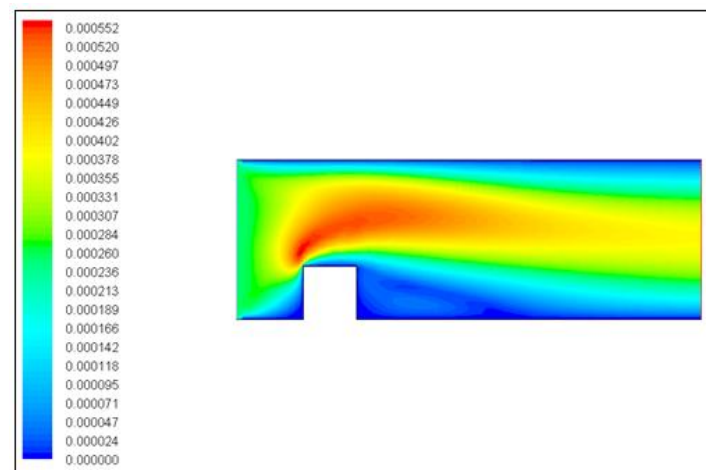
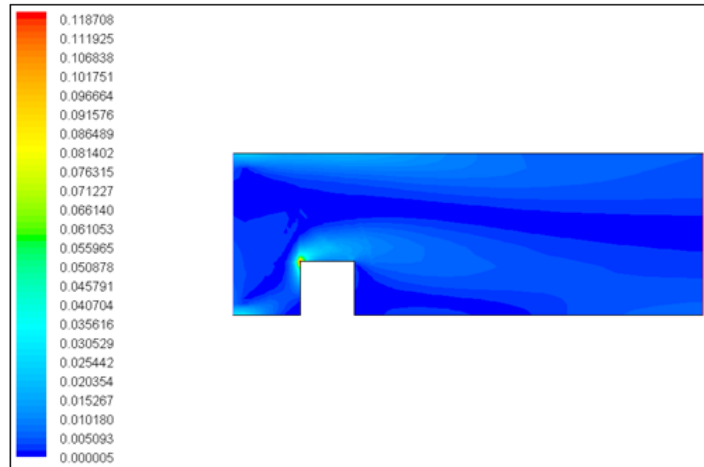
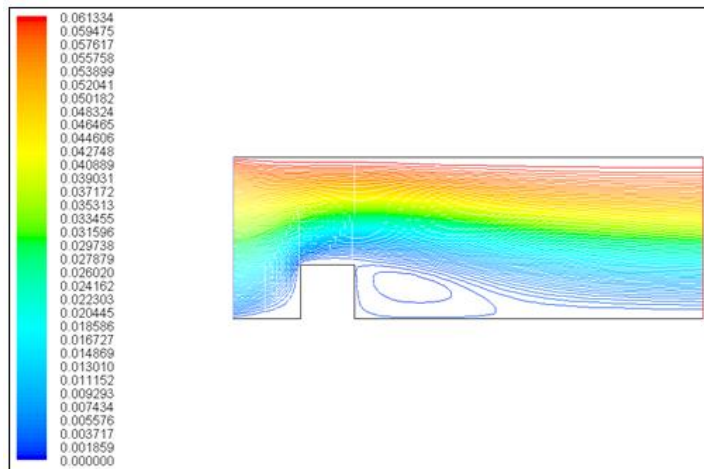
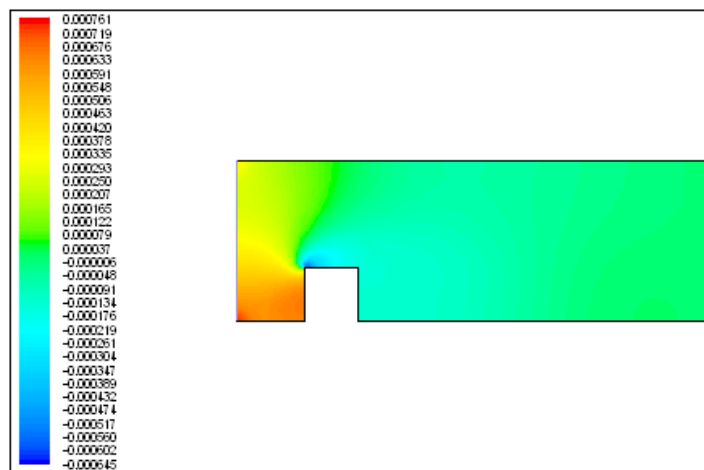
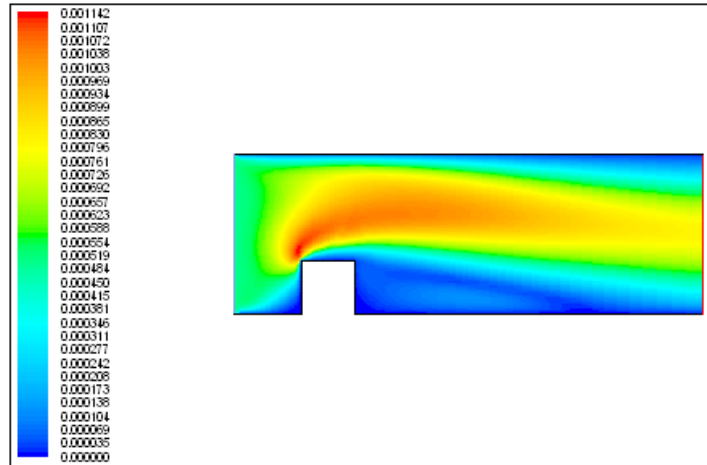
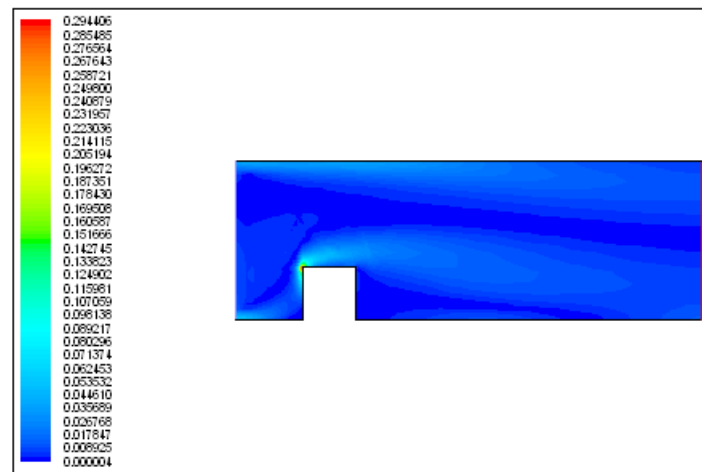
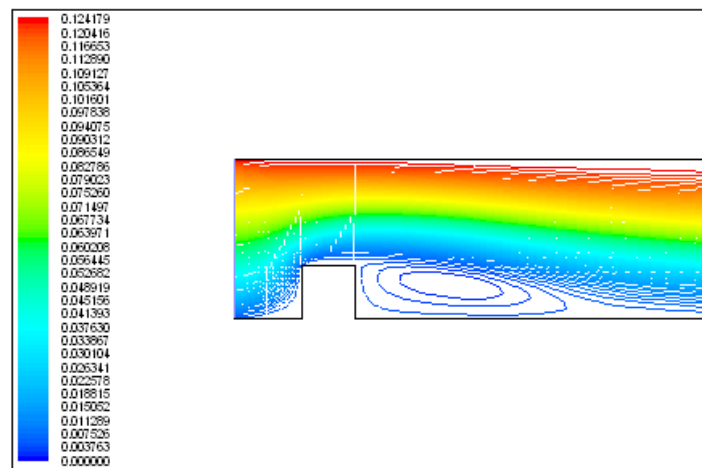
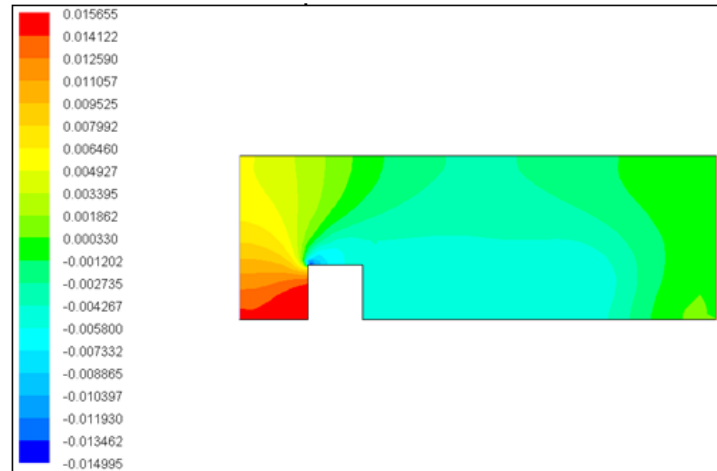
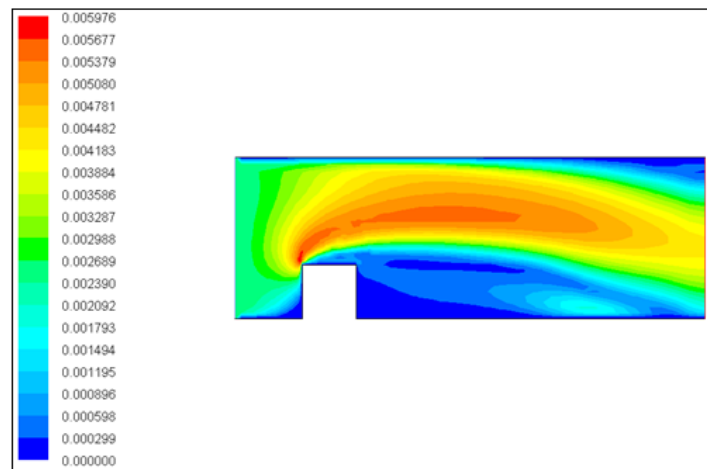
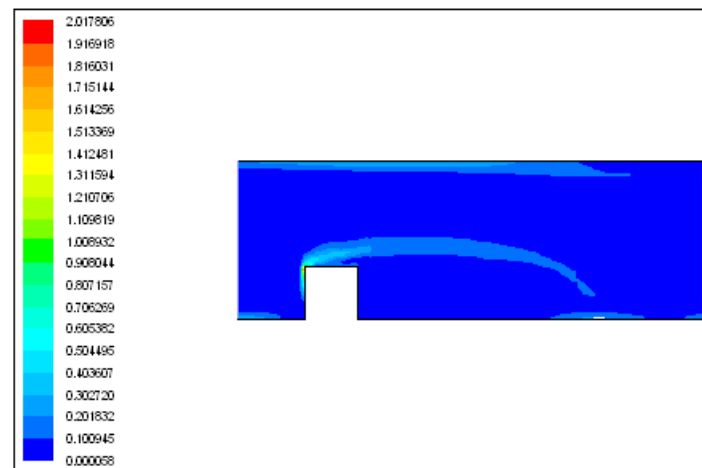
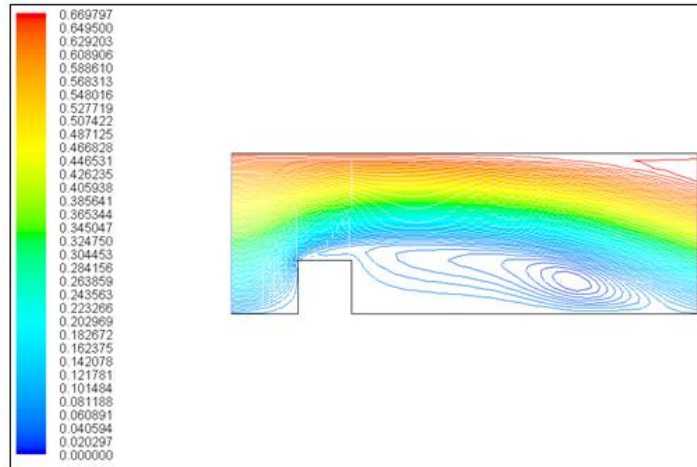
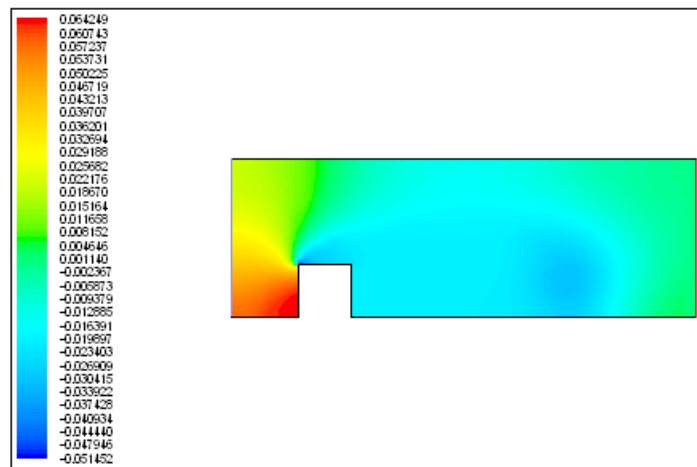
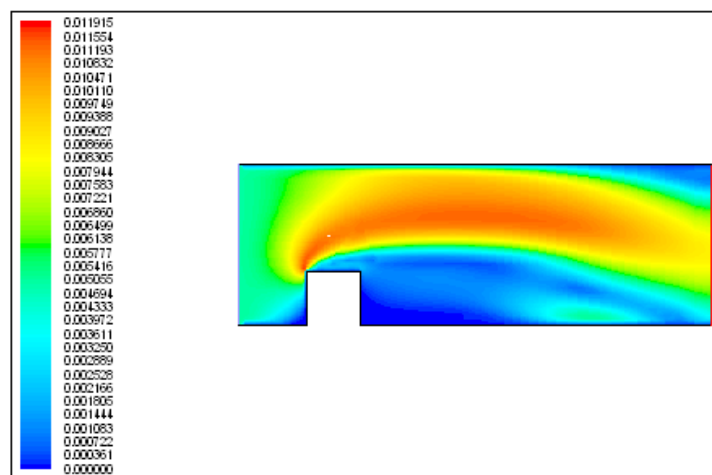


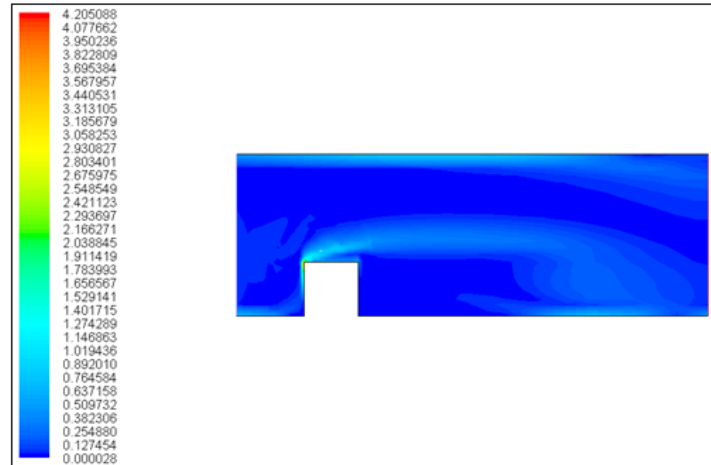
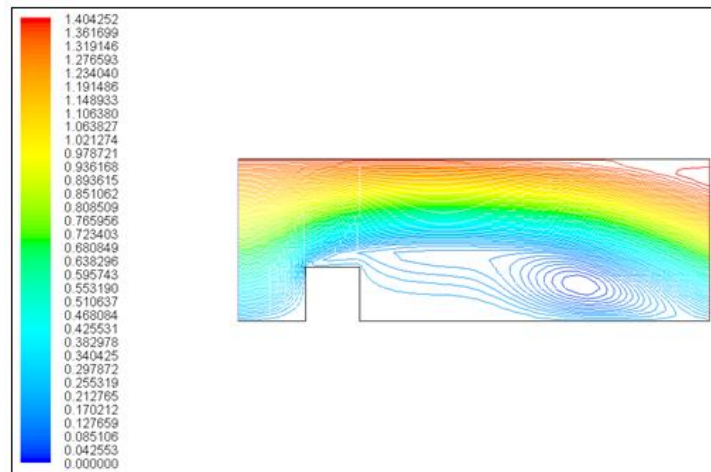
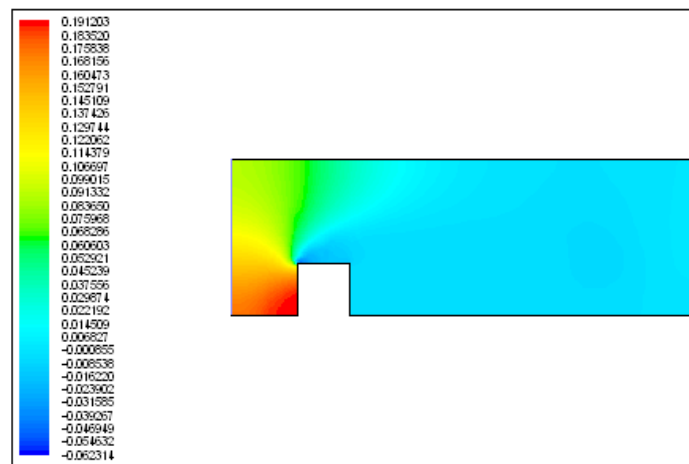
Figure-3.28 Velocity ($Re = 10$)

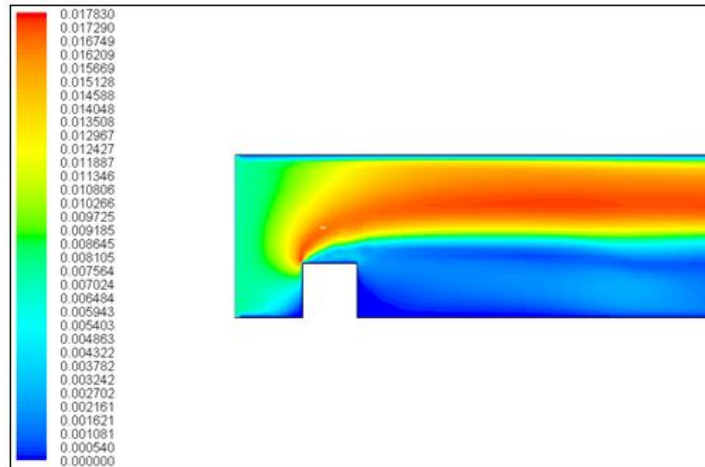
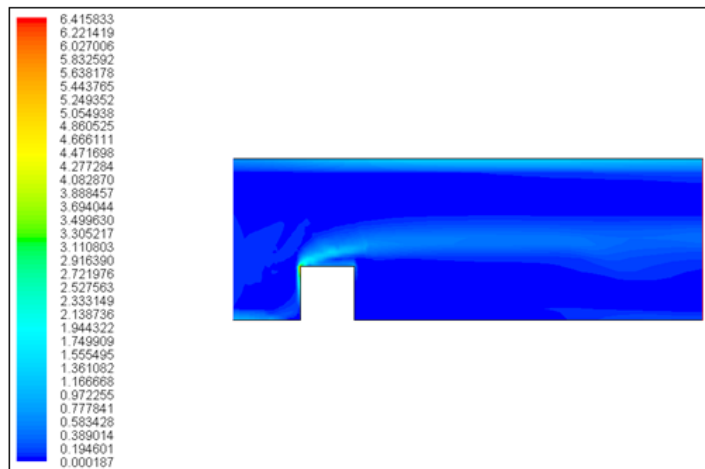
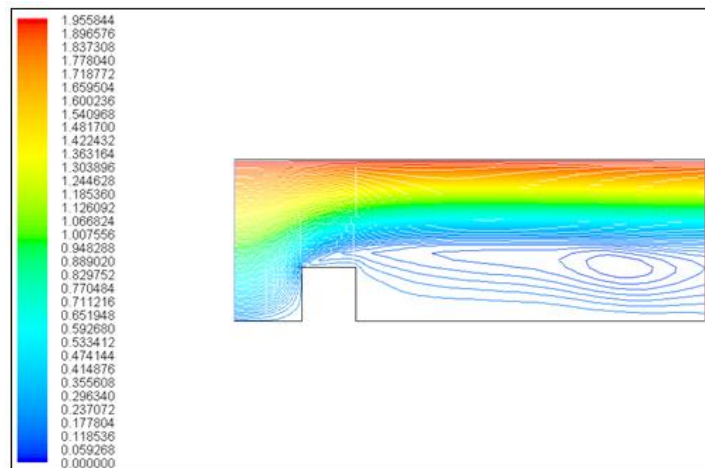
Figure-3.29 Vorticity magnitude (1/s) ($Re = 10$)Figure-3.30 Stream function (Kg/s) ($Re = 10$)Figure-3.31 Static pressure ($Re = 20$)

Figure-3.32 Velocity ($Re = 20$)Figure-3.33 Vorticity magnitude (1/s) ($Re = 20$)Figure-3.34 Stream function (Kg/s) ($Re = 20$)

Figure-3.35 Static pressure ($Re = 100$)Figure-3.36 Velocity ($Re = 100$)Figure-3.37 Vorticity magnitude ($1/s$) ($Re = 100$)

Figure-3.38 Stream function (Kg/s) ($Re = 100$)Figure-3.39 Static pressure ($Re = 200$)Figure-3.40 Velocity ($Re = 200$)

Figure-3.41 Vorticity magnitude (1/s) ($Re = 200$)Figure-3.42 Stream function (Kg/s) ($Re = 200$)Figure-3.43 Static pressure ($Re = 300$)

Figure-3.44 Velocity ($Re = 300$)Figure-3.45 Vorticity magnitude (1/s) ($Re = 300$)Figure-3.46 Stream function (Kg/s) ($Re = 300$)

CHAPTER - 4

PROJECT-2

THE SINGLE PARTICLE SCHRÖDINGER FLUID

As an example of the applications of the second order linear parabolic partial differential equations in quantum mechanics we study in this chapter the solutions of the single particle time dependent Schrödinger wave equation for a nucleon (proton or neutron) which is moving in the average field, created due to the presence of the other $(A - 1)$ particles, where A is the number of nucleons in the nucleus (the mass number). We carry out this treatment in framework of the so-called single particle Schrödinger fluid. The single particle Schrödinger fluid [73] is a concept which is used to describe the motion of a single nucleon in an axially deformed potential of the nucleus. This concept is carried out by a suitable choice of the time- dependent part of the nucleon wave function in the time-dependent Schrödinger equation. This concept can be applied to study the rotational motion of a deformed nucleus.

In this Chapter we carry out the derivations of this concept and accordingly clarify how the moment of inertia of an axially deformed nucleus can be obtained in framework of this concept. As examples for the application of this concept to the calculations of the nuclear moments of inertia we have calculated the cranking-model and the rigid body-model moments of inertia of the even-even axially deformed nuclei in the s-d shell; i.e. the nuclei ^{20}Ne , ^{24}Mg , ^{28}Si , ^{32}S and ^{36}Ar . The variations of the nuclear moments of inertia with respect to the deformation parameter β have been also given in this chapter.

4.1 The Fluid Dynamical Equations

We assume that each nucleon (proton or neutron), with mass m in a nucleus, consisting of A nucleons, is moving in a single-particle potential $V(\mathbf{r}, \alpha(t))$, which is deformed with time t , through its parametric dependence on a classical shape variable $\alpha(t)$. Here, $\alpha(t)$ is assumed to be an externally prescribed function of t . Thus, the Hamiltonian for the present problem is given by [73]

$$H(\mathbf{r}, \mathbf{v}, \alpha(t)) = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, \alpha(t)). \quad (4.1)$$

The operator ∇ in equation (4.1) appeared due to the fact that in, Quantum Mechanics, the operator associated with the particle momentum $m\mathbf{v}$ is given by $-i\hbar\nabla$, where \mathbf{v} is the particle velocity. The single-particle time-dependent wave function $\Psi(\mathbf{r}, \alpha(t), t)$ which satisfies the time-dependent Schrödinger wave equation, that describes the motion of a nucleon, is defined as

$$H(\mathbf{r}, \mathbf{v}, \alpha(t))\Psi(\mathbf{r}, \alpha(t), t) = i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, \alpha(t), t). \quad (4.2)$$

To obtain a fluid dynamical description of the wave function $\Psi(\mathbf{r}, \alpha(t), t)$, we use the polar form of the wave function. We first isolate the explicit time dependence in the form

$$\Psi(\mathbf{r}, \alpha(t), t) = \psi(\mathbf{r}, \alpha(t)) \exp \left\{ -\frac{i}{\hbar} \int_0^t \epsilon(\alpha(t')) dt' \right\}, \quad (4.3)$$

where ϵ is the energy density which depends on the time through the parameter $\alpha(t)$. Then, we write the complex wave function $\psi(\mathbf{r}, \alpha(t))$ in the following polar form

$$\psi(\mathbf{r}, \alpha(t)) = \Phi(\mathbf{r}, \alpha(t)) \exp \left\{ -\frac{iM}{\hbar} S(\mathbf{r}, \alpha(t)) \right\}, \quad (4.4)$$

where $\Phi(\mathbf{r}, \alpha(t))$ and $S(\mathbf{r}, \alpha(t))$ are assumed to be real functions of \mathbf{r} and $\alpha(t)$. Finally, we assume that the function $\Phi(\mathbf{r}, \alpha(t))$ is positive definite. In the case of rotation, the parameter $\alpha(t)$ becomes the angle of rotation, $\theta = \Omega t$, where Ω is the angular velocity.

Substituting equations (4.1), (4.3) and (4.4) into (4.2) we get

$$\begin{aligned} H\Psi(\mathbf{r}, \alpha(t), t) &= \exp \left\{ -\frac{i}{\hbar} \int_0^t \epsilon(\alpha(t')) dt' \right\} \\ &\times \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, \alpha(t)) \right\} \left\{ \Phi(\mathbf{r}, \alpha(t)) \exp \left\{ -\frac{iM}{\hbar} S(\mathbf{r}, \alpha(t)) \right\} \right\} \\ &= i\hbar \frac{\partial}{\partial t} \left[\Phi(\mathbf{r}, \alpha(t)) \exp \left\{ -\frac{iM}{\hbar} S(\mathbf{r}, \alpha(t)) \right\} \exp \left\{ -\frac{i}{\hbar} \int_0^t \epsilon(\alpha(t')) dt' \right\} \right] \end{aligned}$$

So that

$$H\Psi(\mathbf{r}, \alpha(t), t) = \exp \left\{ -\frac{i}{\hbar} \int_0^t \epsilon(\alpha(t')) dt' \right\} \exp \left\{ -\frac{iM}{\hbar} S(\mathbf{r}, \alpha(t)) \right\} \times \left[\epsilon(\alpha(t)) \Phi(\mathbf{r}, \alpha(t)) + M\Phi(\mathbf{r}, \alpha(t)) \frac{\partial}{\partial t} S(\mathbf{r}, \alpha(t)) + i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, \alpha(t)) \right]. \quad (4.5)$$

Hence,

$$\begin{aligned} &\left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, \alpha(t)) \right\} \left\{ \Phi(\mathbf{r}, \alpha(t)) \exp \left\{ -\frac{iM}{\hbar} S(\mathbf{r}, \alpha(t)) \right\} \right\} = \\ &\exp \left\{ -\frac{iM}{\hbar} S(\mathbf{r}, \alpha(t)) \right\} \left[\epsilon(\alpha(t)) \Phi(\mathbf{r}, \alpha(t)) + M\Phi(\mathbf{r}, \alpha(t)) \frac{\partial}{\partial t} S(\mathbf{r}, \alpha(t)) + \right. \\ &\left. i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, \alpha(t)) \right]. \end{aligned} \quad (4.6)$$

But we know that

$$\begin{aligned} \nabla^2 \left(\Phi \exp \left\{ -i \frac{MS}{\hbar} \right\} \right) &= (\nabla^2 \Phi) \exp \left\{ -i \frac{MS}{\hbar} \right\} + \Phi \nabla^2 \left(\exp \left\{ -i \frac{MS}{\hbar} \right\} \right) \\ &+ 2(\nabla \Phi) \cdot \nabla \left(\exp \left\{ -i \frac{MS}{\hbar} \right\} \right), \end{aligned} \quad (4.7)$$

Also we have

$$\nabla \left(\exp \left\{ -i \frac{MS}{\hbar} \right\} \right) = -i \frac{M}{\hbar} (\nabla S) \exp \left\{ -i \frac{MS}{\hbar} \right\},$$

and

$$\begin{aligned}\nabla^2 \left(\exp \left\{ -i \frac{MS}{\hbar} \right\} \right) &= -i \frac{M}{\hbar} \nabla \cdot \left[(\nabla S) \exp \left\{ -i \frac{MS}{\hbar} \right\} \right] \\ &= -i \frac{M}{\hbar} \left[(\nabla^2 S) \exp \left\{ -i \frac{MS}{\hbar} \right\} - i \frac{M}{\hbar} (\nabla S) \cdot (\nabla) \exp \left\{ -i \frac{MS}{\hbar} \right\} \right] \\ &= -i \frac{M}{\hbar} (\nabla^2 S) \exp \left\{ -i \frac{MS}{\hbar} \right\} - \frac{M^2}{\hbar^2} (\nabla S) \cdot (\nabla S) \exp \left\{ -i \frac{MS}{\hbar} \right\}.\end{aligned}$$

Substituting from the above results into equation (4.6) we get

$$\begin{aligned}i\hbar \frac{\partial \Phi}{\partial t} \exp \left\{ -i \frac{MS}{\hbar} \right\} + M\Phi \frac{\partial S}{\partial t} \exp \left\{ -i \frac{MS}{\hbar} \right\} + \epsilon \Phi \exp \left\{ -i \frac{MS}{\hbar} \right\} &= -\frac{\hbar^2}{2M} \exp \left\{ -i \frac{MS}{\hbar} \right\} \nabla^2 \Phi + \\ \frac{M}{2} \Phi \exp \left\{ -i \frac{MS}{\hbar} \right\} (\nabla S) \cdot (\nabla S) + \frac{i\hbar}{2} \Phi \exp \left\{ -i \frac{MS}{\hbar} \right\} (\nabla^2 S) &+ i\hbar (\nabla \Phi) \cdot (\nabla S) \exp \left\{ -i \frac{MS}{\hbar} \right\} + \\ V\Phi \exp \left\{ -i \frac{MS}{\hbar} \right\}.\end{aligned}\quad (4.8)$$

Dividing all the terms by $\exp \left\{ -i \frac{MS}{\hbar} \right\}$ we get

$$i\hbar \frac{\partial \Phi}{\partial t} + M\Phi \frac{\partial S}{\partial t} + \epsilon \Phi = -\frac{\hbar^2}{2M} \nabla^2 \Phi + \frac{i\hbar}{2} \Phi (\nabla^2 S) + \frac{M}{2} \Phi (\nabla S)^2 + i\hbar (\nabla \Phi) \cdot (\nabla S) + V\Phi. \quad (4.9)$$

Hence,

$$i \left[\hbar \frac{\partial \Phi}{\partial t} - \frac{\hbar}{2} \Phi (\nabla^2 S) - \hbar (\nabla \Phi) \cdot (\nabla S) \right] + M\Phi \frac{\partial S}{\partial t} + \epsilon \Phi - H\Phi - \frac{M}{2} \Phi (\nabla S) \cdot (\nabla S) = 0. \quad (4.10)$$

This yields, from its real and imaginary parts, a pair of coupled equations for Φ and S as follows:

$$\left[H - M \left(\frac{\partial S}{\partial t} - \frac{1}{2} \nabla S \cdot \nabla S \right) \right] \Phi = \epsilon \Phi. \quad (4.11)$$

and

$$\frac{1}{2} \Phi (\nabla^2 S) + (\nabla \Phi) \cdot (\nabla S) = \frac{\partial \Phi}{\partial t}. \quad (4.12)$$

We may call equation (4.11) modified Schrödinger equation because it differs from the usual time-independent Schrödinger equation $H\Phi = \epsilon\Phi$ by an added term which we refer to as the “dynamical modification potential”

$$V_{dyn} = -M \left[\frac{\partial S}{\partial t} - \frac{1}{2} (\nabla S) \cdot (\nabla S) \right]. \quad (4.13)$$

4.2 Interpretation of the Probability as a Fluid Continuity Equation

When we identify the probability density of the single particle as the square of the amplitude $|\Phi|^2$ and recognize that equation (4.12), when multiplied by 2Φ , gives

$$\Phi^2 \nabla^2 S + \nabla \Phi^2 \cdot \nabla S = \frac{\partial \Phi^2}{\partial t}, \quad (4.14)$$

then, we will obtain two equations, the first is

$$\rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = -\frac{\partial \rho}{\partial t}, \quad (4.15)$$

where \mathbf{v} is the irrotational velocity and ρ is the density. It is the well-known equation of continuity in fluid mechanics. It can be rewritten in the form:

$$\nabla \cdot (\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t}, \quad (4.16)$$

where $\rho = \Phi^2$ and $\mathbf{v} = -\nabla S$.

The second equation is

$$(H + V_{dyn})\Phi = \epsilon\Phi, \quad (4.17)$$

which is a modified Schrödinger equation with

$$V_{dyn} = -M \left(\frac{\partial S}{\partial t} - \frac{1}{2} \mathbf{v}^2 \right). \quad (4.18)$$

4.3 The Relationship Between the Velocity Field and the Current

From equation (4.4) we can easily obtain

$$S = \frac{i\hbar}{2M} \ln \left(\frac{\psi}{\psi^*} \right). \quad (4.19)$$

We know that,

$$\psi = \Phi \exp \left\{ -i \frac{MS}{\hbar} \right\}, \quad \psi^* = \Phi \exp \left\{ i \frac{MS}{\hbar} \right\},$$

and

$$\mathbf{v} = -\nabla S = -\frac{i\hbar}{2M} \nabla \left(\ln \left(\frac{\psi}{\psi^*} \right) \right) = -\frac{i\hbar}{2M} [\nabla(\ln(\psi)) - \nabla(\ln(\psi^*))] = \frac{i\hbar}{2M} \left[\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi} \right].$$

Therefore,

$$\mathbf{v} = \frac{i\hbar}{2M|\psi|^2} [\psi \nabla \psi^* - \psi^* \nabla \psi].$$

The current of the single particle state is defined by

$$\mathbf{j} = \rho \mathbf{v} = \frac{i\hbar \rho}{2M|\psi|^2} [\psi \nabla \psi^* - \psi^* \nabla \psi].$$

Putting $\rho = |\Phi|^2$, we get

$$\mathbf{j} = \frac{i\hbar}{2M} \frac{|\Phi|^2}{|\psi|^2} [\psi \nabla \psi^* - \psi^* \nabla \psi].$$

Since,

$$|\psi|^2 = |\Phi|^2 \left| e^{-\frac{iMS}{\hbar}} \right|^2 = |\Phi|^2,$$

we finally get

$$\mathbf{j} = \frac{i\hbar}{2M} [\psi \nabla \psi^* - \psi^* \nabla \psi]. \quad (4.20)$$

4.4 Euler's Equation and the Equation of State

Euler's equation for the non-viscous fluid flow is given by

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\rho}, \quad (4.21)$$

where P is the pressure on the fluid at a point $P(\mathbf{r})$ at an instant of time t . For an ideal fluid, ∇P is related to the enthalpy per unit mass, w , of the fluid by the following manner

$$\frac{\nabla P}{\rho} = \nabla w. \quad (4.22)$$

Therefore, Euler's equation can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla w. \quad (4.23)$$

After integration and using $\mathbf{v} = -\nabla S$ we get

$$\frac{\partial S}{\partial t} - \frac{1}{2} v^2 = w. \quad (4.24)$$

Using also

$$v^2 = (\nabla S)^2,$$

we get

$$\frac{\partial S}{\partial t} - \frac{1}{2} (\nabla S)^2 = w, \quad (4.25)$$

where S is the velocity potential for \mathbf{v} , ($\mathbf{v} = -\nabla S$) and the constant of integration in equation (4.24) chosen here to be zero. Therefore, we can write

$$V_{dyn} = -M \left[\frac{\partial S}{\partial t} - \frac{1}{2} (\nabla S)^2 \right], \quad (4.26)$$

$$V_{dyn} = -Mw, \quad (4.27)$$

and then the modified Schrödinger equation takes the

$$(H - Mw)\Phi = \epsilon\Phi, \quad (4.28)$$

where w is now the “enthalpy” of the single-particle Schrödinger fluid.

Hence, we have a set of fluid dynamical equations completely analogous to those which describe a classical fluid. This set consists of the continuity equation (4.15), the Euler equation (4.25), and an equation of state (4.28). By derivation, their content is precisely that of the original time-dependent Schrödinger equation. Hill and Wheeler [116] assumed that the single-particle Schrödinger fluid is irrotational and implicitly incompressible flow. The present formulation is specifically not restricted to incompressible flows but allows also irrotational but compressible.

The description of the density $|\psi|^2$ as a classical fluid implies that we are assigning labels to each “mass element” $|\psi|^2\Delta x\Delta y\Delta z$, and considering its motion in time, as described by the velocity field \mathbf{v} . However, in quantum mechanics, the quantity $|\psi|^2\Delta x\Delta y\Delta z$ is interpreted as the probability of finding the nucleon in the volume element $\Delta x\Delta y\Delta z$.

In addition to the irrotational velocity \mathbf{v} , which is a result from the fluid dynamical equation, other velocity fields which satisfy the continuity equation of the Schrödinger equation occur. Among these velocity fields are [73] the incompressible velocity field, the regular velocity field, the geometric velocity field, and the rigid body velocity field. For rotations, the rigid-body velocity field \mathbf{v}_{rig} is defined as

$$\mathbf{v}_{rig} = \boldsymbol{\Omega} \times \mathbf{r}. \quad (4.29)$$

It is seen that this velocity field is incompressible, regular, and also of geometric type.

4.5 The Collective Kinetic Energy for the Entire Nucleus

In the adiabatic approximation, where $\frac{\partial\alpha}{\partial t} \rightarrow 0$, that is the angle of rotation θ is constant of the time, the collective kinetic energy of a nucleon in the nucleus is given by [73]

$$T_K = \frac{1}{2} \int \rho \mathbf{v}_K \cdot (\boldsymbol{\Omega} \times \mathbf{r}) dr. \quad (4.30)$$

and the collective kinetic energy T of the nucleus is given by

$$T = \frac{1}{2} M \int \rho_T \mathbf{v}_T \cdot (\boldsymbol{\Omega} \times \mathbf{r}) dr, \quad (4.31)$$

where ρ_T is the total density distribution of the nucleus and \mathbf{v}_T is the total velocity field,

$$\mathbf{v}_T = \frac{\sum_{occ} \rho_K \mathbf{v}_K}{\sum_{occ} \rho_K}. \quad (4.32)$$

The collective kinetic energy for the entire nucleus, as an integral of a density weighted quadratic form in velocities, conforms to the classical structure of a continuum kinetic energy, but involves two distinct velocity fields, rather than simply the square of a single velocity field. The occurrence of these two distinct velocity fields reflects the two essential aspects of the cranking motion:

- (i) The rotation of the potential well, which can (just as can the density's variation in time for pure rotation) be described as the regular velocity field $\mathbf{\Omega} \times \mathbf{r}$.
- (ii) The response of the individual particle to the motion of the potential, described by \mathbf{v}_K .

This structure seems eminently natural, and perhaps, sufficiently general to encompass a wide variety of the kinetic energy forms.

4.6 Single Particle in the Harmonic Oscillator Potential

The single particle oscillator wave functions are taken in the form of products of three one-dimensional oscillator functions of the form [QM]

$$u_{n_x} u_{n_y} u_{n_z} = u_{n_x}(\zeta) u_{n_y}(\eta) u_{n_x}(\xi), \quad (4.33)$$

where

$$u_{n_z}(\xi) = \frac{1}{\sqrt{2^{n_z} n_z!}} \left(\frac{m\omega_z}{\pi\hbar} \right)^{\frac{1}{4}} H_{n_z}(\xi) \exp\left(-\frac{1}{2}\xi^2\right). \quad (4.34)$$

Similar equations hold for $u_x(\xi)$ (and $u_{n_y}(\eta)$). In (4.34) $H_{n_z}(\xi)$ is the Hermite polynomial, and the dimensionless variables are defined as

$$(\zeta, \eta, \xi) = \left(\frac{\sqrt{m\omega_x}}{\hbar} x, \frac{\sqrt{m\omega_y}}{\hbar} y, \frac{\sqrt{m\omega_z}}{\hbar} z \right). \quad (4.35)$$

Here, we restrict the discussion to the axially symmetric geometry for simplicity. Hence, $\omega_x = \omega_y$ and the intrinsic energy of the single particle state is given by

$$E_{n_x n_y n_z} = \hbar\omega_x(n_x + n_y + 1) + \hbar\omega_z(n_z + 1). \quad (4.36)$$

Using the perturbation theory, we can calculate the cranking correction to the wave function [73] explicitly, and the result is

$$\mu_k = \Omega \sum_{j \neq k} \frac{\langle j | \mathcal{L} | k \rangle}{\epsilon_j - \epsilon_k} u_j, \quad (4.37)$$

where

$$\mu_{n_x n_y n_z} = \mu_{n_x}(\xi) \mu_{n_y n_z}(\eta, \zeta) = -\frac{\Omega u_{n_x}}{2\sqrt{\omega_y \omega_z}} \left\{ \begin{array}{l} \sigma \sqrt{n_y n_z} u_{n_y-1} u_{n_z-1} \\ + \frac{1}{\sigma} \sqrt{n_y (n_z + 1)} u_{n_y-1} u_{n_z+1} \\ + \frac{1}{\sigma} \sqrt{n_z (n_y + 1)} u_{n_y+1} u_{n_z-1} \\ + \sigma \sqrt{(n_y + 1)(n_z + 1)} u_{n_y+1} u_{n_z+1} \end{array} \right\} \quad (4.38)$$

Here μ_k is the first-order time-dependent perturbation correction for rotation about the z-axis, the functions with n_x, n_y and n_z subscripts, with arguments ξ, η and ζ , respectively and

$$\sigma = \frac{\omega_y - \omega_z}{\omega_y + \omega_z}, \quad (4.39)$$

is a measure of the deformation of the potential.

We introduce one single parameter of deformation δ given by [117]

$$\omega_z^2 = \omega_0^2 \left(1 - \frac{4}{3} \delta\right), \quad (4.40)$$

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(1 + \frac{2}{3} \delta\right). \quad (4.41)$$

The condition of constant volume of the nucleus leads to

$$\omega_x \omega_y \omega_z = \text{const.} \quad (4.42)$$

Keeping this condition in the general case together with (4.40) and (4.41), ω_0 must depend on δ in the following way [117]

$$\omega_0(\delta) = \omega_0^0 \left\{1 - \frac{12}{9} \delta^2 - \frac{16}{27} \delta^3\right\}^{-\frac{1}{6}},$$

where ω_0^0 is the value of $\omega_0(\delta)$ for $\delta = 0$. $\hbar \omega_0^0$ is known as the non-deformed oscillator parameter. This parameter can be calculated from the values of the total number of protons in the nucleus Z , the number of neutrons N and the mass number A as follows [69]

$$\hbar \omega_0^0 = \frac{38.6A^{-\frac{1}{3}}}{\left[1 + \frac{1.646}{A} - \frac{0.191(N-Z)}{A}\right]^2}.$$

The deformation parameter δ is related to the well-known deformation parameter β by

$$\delta = \frac{3}{2} \sqrt{\frac{5}{4\pi}} \beta = 0.95\beta. \quad (4.43)$$

The parameter β is allowed to vary in the range $-0.5 \leq \beta \leq 0.5$.

4.7 Rigid (Equilibrium) Oscillator Moments of Inertia from Fluid Dynamical Viewpoint

We now examine the cranking moment of inertia in terms of the velocity fields. Bohr and Mottelson [118] show that for harmonic oscillator case at the equilibrium deformation, where

$$\frac{d}{d\delta} \sum_{i=1} \left(E_{n_x n_y n_z} \right)_i = 0, \quad (4.44)$$

the cranking moment of inertia is identically equal to the rigid moment of inertia:

$$\mathfrak{J}_{cr} = \mathfrak{J}_{rig} = \sum_{i=1} m(y_i^2 + z_i^2). \quad (4.45)$$

In terms of expression (4.31) involving the velocity fields, this result asserts the equality of the collective kinetic energy of the Schrödinger fluid and that of rigidly rotating classical fluid

$$\frac{m}{2} \int \rho_T \mathbf{v}_T \cdot (\boldsymbol{\Omega} \times \mathbf{r}) d\tau = \frac{1}{2} \mathfrak{J}_{rig} \Omega^2 = \frac{m}{2} \int \rho_T (\boldsymbol{\Omega} \times \mathbf{r})^2 d\tau, \quad (4.46)$$

at the equilibrium deformation. We emphasize that equations (4.45) and (4.46) hold for any number of nucleons occupying any set of single particle harmonic oscillator states at the deformation defined by equilibrium condition (4.44). In particular, it holds for a one particle state. For this case, equation (4.46) becomes

$$\frac{m}{2} \int \rho_K \mathbf{v}_K \cdot (\boldsymbol{\Omega} \times \mathbf{r}) d\tau = \frac{m}{2} \int \rho_K (\boldsymbol{\Omega} \times \mathbf{r})^2 d\tau, \quad (4.47)$$

at the equilibrium deformation of the single particle state

$$|i\rangle \equiv |n_x n_y n_z\rangle.$$

Equation (4.47) is a remarkable identity. The scalar product of \mathbf{v}_K and $(\boldsymbol{\Omega} \times \mathbf{r})$ which occurs on the left side is replaced on the right side, by the absolute square of $(\boldsymbol{\Omega} \times \mathbf{r})$.

It forces one to inquire whether the irrotational field \mathbf{v}_K is equal to $(\boldsymbol{\Omega} \times \mathbf{r})$. The answer, of course, is “no“. For, \mathbf{v}_K posses compressible line vortices. It can, therefore, never be equal to velocity field for rigid rotation $\mathbf{v}_{rig} = \boldsymbol{\Omega} \times \mathbf{r}$, which has no singularity and is everywhere incompressible and rotational.

Despite this qualitative difference between \mathbf{v}_K and the other velocity in equation (4.47), this shows that, as regards their effects under the integral upon the overall kinetic energy (or the internal parameter), these two velocity fields are equivalent at the equilibrium deformation.

4.8 Cranking Moments and Rigid-body Moments of Inertia

We note that the cranking moment of inertia \mathfrak{I}_{cr} and the rigid moment of inertia \mathfrak{I}_{rig} are equal only when the harmonic oscillator is at the equilibrium deformation. At other deformations, they can, and do, deviate substantially from one another [73].

The following-expressions for the cranking moment of inertia. and the rigid moment of inertia \mathfrak{I}_{rig} as a function of q are given:

$$\mathfrak{I}_{cr} = \frac{E}{\omega_0^2} \left(\frac{1}{6+2\sigma} \right) \left(\frac{1+\sigma}{1-\sigma} \right)^{\frac{1}{3}} \left[\sigma^2(1+q) + \frac{1}{\sigma}(1-q) \right], \quad (4.48)$$

$$\mathfrak{I}_{rig} = \frac{E}{\omega_0^2} \left(\frac{1}{6+2\sigma} \right) \left(\frac{1+\sigma}{1-\sigma} \right)^{\frac{1}{3}} [(1+q) + \sigma(1-q)], \quad (4.49)$$

where E is the total nuclear ground-state energy

$$E = \sum_{occ} [\hbar\omega_x(n_x + n_y + 1) + \hbar\omega_z(n_z + 1)], \quad (4.50)$$

and q is the ratio of the summed single particle quanta in the y - and z -directions

$$q = \frac{\sum_{occ}(n_y+1)}{\sum_{occ}(n_z+1)}. \quad (4.51)$$

q is known as the anisotropy of the configuration. In equations (4.48) and (4.49) the deformation of the potential, σ is defined by [73]

$$\sigma = \frac{(\omega_y - \omega_z)}{(\omega_y + \omega_z)}.$$

4.10 Results and Conclusions

We have constructed the ground states of the five nuclei ^{20}Ne , ^{24}Mg , ^{28}Si , ^{32}S and ^{36}Ar , which are all even-even (number of protons Z is even and number of neutrons N is also even) deformed nuclei. Accordingly, the single particle states in each nucleus are filled with the corresponding wave functions. As a result, \mathfrak{I}_{cr} and \mathfrak{I}_{rig} are calculated for each nucleus. Finally, the corresponding reciprocal moments $\frac{\hbar^2}{2\mathfrak{I}_{crank}}$ and $\frac{\hbar^2}{2\mathfrak{I}_{rigid}}$ are calculated.

In Tables-4.1, 4.2, 4.3, 4.4 and 4.5 we present the calculated values of the reciprocal moments of inertia according to the cranking model and the rigid-body model as functions of the deformation parameter β for the nuclei ^{20}Ne , ^{24}Mg , ^{28}Si , ^{32}S and ^{36}Ar , respectively. The values of the non-deformed oscillator parameter $\hbar\omega_0^0$ for the five nuclei are also given in these tables.

Table-4.1 Reciprocal moments of inertia of ^{20}Ne as functions of β (A=20 Z=10 N=10 $\hbar\omega_0^0 = 11.881$)

β	$\frac{\hbar^2}{2\mathfrak{I}_{crank}}$	$\frac{\hbar^2}{2\mathfrak{I}_{rigid}}$
-0.500	590.624000	362.826800
-0.490	578.555000	360.987800
-0.480	566.509800	359.202700
-0.470	554.485800	357.469200
-0.460	542.480800	355.785100
-0.450	530.493500	354.148500
-0.440	518.521700	352.557500
-0.430	506.564000	351.010200
-0.420	494.619600	349.505100
-0.410	482.687500	348.040700
-0.400	470.766700	346.615500
-0.390	458.856800	345.228100
-0.380	446.956900	343.877200
-0.370	435.067000	342.561700
-0.360	423.186600	341.280400
-0.350	411.315400	340.032200
-0.340	399.453400	338.816100
-0.330	387.600400	337.631200
-0.320	375.756400	336.476600
-0.310	363.921300	335.351300
-0.300	352.095100	334.254500
-0.290	340.277800	333.185600
-0.280	328.469500	332.143700
-0.270	316.670000	331.128200
-0.260	304.879600	330.138400
-0.250	293.098100	329.173700
-0.240	281.325300	328.233400
-0.230	269.561500	327.317100
-0.220	257.806200	326.424100
-0.210	246.059400	325.554000
-0.200	234.320600	324.706300

Table-4.1 (Continued)

-0.190	222.589700	323.880400
-0.180	210.866300	323.076000
-0.170	199.149900	322.292700
-0.160	187.439700	321.530000
-0.150	175.735300	320.787700
-0.140	164.035800	320.065100
-0.130	152.340300	319.362200
-0.120	140.647800	318.678400
-0.110	128.957100	318.013700
-0.100	117.267000	317.367600
-0.090	105.576100	316.739800
-0.080	93.882770	316.130000
-0.070	82.185250	315.538200
-0.060	70.481680	314.963900
-0.050	58.769890	314.407000
-0.040	47.047620	313.867300
-0.030	35.312380	313.344500
-0.020	23.561350	312.838600
-0.010	11.791780	312.349200
0.000	0.000253	311.876300
0.010	11.816600	311.419600
0.020	23.662550	310.979100
0.030	35.541660	310.554600
0.040	47.458230	310.146000
0.050	59.417000	309.753200
0.060	71.423040	309.375900
0.070	83.481920	309.014400
0.080	95.599380	308.668300
0.090	107.781800	308.337600
0.100	120.036200	308.022200
0.110	132.369800	307.722200
0.120	144.790500	307.437400
0.130	157.306800	307.167800
0.140	169.928100	306.913400
0.150	182.664400	306.674100

Table-4.1 (Continued)

0.160	195.526400	306.450000
0.170	208.525700	306.241000
0.180	221.675000	306.047100
0.190	234.987900	305.868300
0.200	248.479300	305.704700
0.210	262.165400	305.556300
0.220	276.063500	305.423100
0.230	290.193100	305.305100
0.240	304.574800	305.202500
0.250	319.231500	305.115300
0.260	334.188400	305.043400
0.270	349.472900	304.987100
0.280	365.115300	304.946400
0.290	381.149000	304.921200
0.300	397.611100	304.911900
0.310	414.542400	304.918400
0.320	431.988800	304.940900
0.330	450.000900	304.979400
0.340	468.636000	305.034200
0.350	487.957800	305.105200
0.360	508.038600	305.192600
0.370	528.960000	305.296600
0.380	550.814800	305.417200
0.390	573.708800	305.554600
0.400	597.763500	305.708900
0.410	623.119100	305.880200
0.420	649.937400	306.068500
0.430	678.407800	306.274100
0.440	708.752200	306.497000
0.450	741.233000	306.737100
0.460	776.162300	306.994600
0.470	813.914600	307.269400
0.480	854.943900	307.561500
0.490	899.805400	307.870800
0.500	949.185500	308.197000

Table-4.2 Reciprocal moments of inertia of ^{24}Mg as functions of β

$$(A = 24 \quad Z = 12 \quad N = 12 \quad \hbar\omega_0^0 = 11.547)$$

β	$\frac{\hbar^2}{2\mathfrak{I}_{crank}}$	$\frac{\hbar^2}{2\mathfrak{I}_{rigid}}$
-0.500	267.196400	251.572900
-0.490	261.279800	250.293900
-0.480	255.405500	249.052600
-0.470	249.571500	247.847600
-0.460	243.775800	246.677200
-0.450	238.016700	245.540200
-0.440	232.292400	244.435200
-0.430	226.601600	243.361100
-0.420	220.942600	242.316400
-0.410	215.314300	241.300400
-0.400	209.715400	240.312000
-0.390	204.144700	239.350100
-0.380	198.601100	238.414000
-0.370	193.083500	237.502700
-0.360	187.591200	236.615600
-0.350	182.122900	235.751700
-0.340	176.677900	234.910500
-0.330	171.255400	234.091200
-0.320	165.854600	233.293300
-0.310	160.474600	232.516200
-0.300	155.114600	231.759000
-0.290	149.774000	231.021600
-0.280	144.452100	230.303300
-0.270	139.148000	229.603600
-0.260	133.861200	228.922100
-0.250	128.591000	228.258400
-0.240	123.336600	227.611900
-0.230	118.097200	226.982400
-0.220	112.872500	226.369500
-0.210	107.661500	225.772700
-0.200	102.463500	225.191800

Table-4.2 (Continued)

-0.190	97.277980	224.626400
-0.180	92.104100	224.076300
-0.170	86.941120	223.541100
-0.160	81.788280	223.020700
-0.150	76.644830	222.514600
-0.140	71.509990	222.022700
-0.130	66.382940	221.544700
-0.120	61.262760	221.080500
-0.110	56.148630	220.629700
-0.100	51.039660	220.192300
-0.090	45.934850	219.767900
-0.080	40.833200	219.356500
-0.070	35.733690	218.957900
-0.060	30.635300	218.571900
-0.050	25.536820	218.198400
-0.040	20.437110	217.837200
-0.030	15.334880	217.488100
-0.020	10.228900	217.151100
-0.010	5.117772	216.826000
0.000	.000111	216.512900
0.010	5.125653	216.211500
0.020	10.261120	215.921700
0.030	15.407900	215.643500
0.040	20.567820	215.376800
0.050	25.742720	215.121600
0.060	30.934540	214.877700
0.070	36.145430	214.645100
0.080	41.377550	214.423800
0.090	46.633270	214.213800
0.100	51.915050	214.014900
0.110	57.225560	213.827200
0.120	62.567580	213.650700
0.130	67.944070	213.485400
0.140	73.358280	213.331100
0.150	78.813510	213.188100

Table-4.2 (Continued)

0.160	84.313480	213.056200
0.170	89.861960	212.935400
0.180	95.463140	212.825900
0.190	101.121400	212.727500
0.200	106.841600	212.640500
0.210	112.628800	212.564800
0.220	118.488500	212.500300
0.230	124.426700	212.447300
0.240	130.449600	212.405800
0.250	136.564400	212.375800
0.260	142.778400	212.357500
0.270	149.099800	212.350800
0.280	155.537400	212.355900
0.290	162.101000	212.372900
0.300	168.801000	212.402000
0.310	175.649000	212.443000
0.320	182.657500	212.496400
0.330	189.840600	212.562100
0.340	197.213400	212.640300
0.350	204.793000	212.731100
0.360	212.598000	212.834700
0.370	220.649200	212.951100
0.380	228.970100	213.080700
0.390	237.586500	213.223400
0.400	246.527600	213.379500
0.410	255.826400	213.549200
0.420	265.519900	213.732500
0.430	275.650700	213.929700
0.440	286.266900	214.141000
0.450	297.424000	214.366300
0.460	309.185700	214.606000
0.470	321.626300	214.860100
0.480	334.832000	215.128600
0.490	348.904500	215.411900
0.500	363.963600	215.709700

Table-4.3 Reciprocal moments of inertia of ^{28}Si as functions of β

$$(A = 28 \quad Z = 14 \quad N = 14 \quad \hbar\omega_0^0 = 11.218)$$

β	$\frac{\hbar^2}{2\mathfrak{I}_{crank}}$	$\frac{\hbar^2}{2\mathfrak{I}_{rigid}}$
-0.500	541.606400	231.846400
-0.490	531.513200	230.638700
-0.480	521.376400	229.465800
-0.470	511.195100	228.326000
-0.460	500.968800	227.218000
-0.450	490.697500	226.140600
-0.440	480.380900	225.092500
-0.430	470.019600	224.072600
-0.420	459.613800	223.079800
-0.410	449.164700	222.113200
-0.400	438.672700	221.171700
-0.390	428.139100	220.254600
-0.380	417.565000	219.360900
-0.370	406.952100	218.490000
-0.360	396.301500	217.641000
-0.350	385.614600	216.813200
-0.340	374.893400	216.006100
-0.330	364.139500	215.218900
-0.320	353.354300	214.451200
-0.310	342.539800	213.702200
-0.300	331.697800	212.971500
-0.290	320.830000	212.258700
-0.280	309.938100	211.563100
-0.270	299.024100	210.884500
-0.260	288.089500	210.222200
-0.250	277.136100	209.576000
-0.240	266.165500	208.945400
-0.230	255.179200	208.330100
-0.220	244.178700	207.729600
-0.210	233.165700	207.143800
-0.200	222.141200	206.572200

Table-4.3 (Continued)

-0.190	211.106300	206.014500
-0.180	200.062600	205.470500
-0.170	189.010500	204.939800
-0.160	177.951200	204.422300
-0.150	166.885300	203.917600
-0.140	155.813400	203.425600
-0.130	144.735500	202.945900
-0.120	133.652400	202.478400
-0.110	122.563600	202.022900
-0.100	111.469000	201.579200
-0.090	100.368400	201.147000
-0.080	89.261000	200.726200
-0.070	78.145960	200.316800
-0.060	67.022240	199.918300
-0.050	55.888310	199.530800
-0.040	44.742810	199.154000
-0.030	33.583620	198.788000
-0.020	22.408460	198.432300
-0.010	11.214980	198.087100
0.000	0.000246	197.752200
0.010	11.239140	197.427400
0.020	22.506650	197.112700
0.030	33.806350	196.807900
0.040	45.142690	196.513100
0.050	56.520660	196.228000
0.060	67.945700	195.952600
0.070	79.423650	195.686800
0.080	90.961090	195.430600
0.090	102.565200	195.183900
0.100	114.243700	194.946700
0.110	126.005100	194.718900
0.120	137.858700	194.500400
0.130	149.814800	194.291200
0.140	161.884200	194.091300
0.150	174.079100	193.900600

Table-4.3 (Continued)

0.160	186.412700	193.719200
0.170	198.899400	193.546900
0.180	211.555000	193.383800
0.190	224.396600	193.229900
0.200	237.443300	193.085100
0.210	250.716000	192.949500
0.220	264.237200	192.823100
0.230	278.032100	192.705700
0.240	292.128900	192.597500
0.250	306.558000	192.498500
0.260	321.353800	192.408700
0.270	336.553800	192.328100
0.280	352.200700	192.256700
0.290	368.341700	192.194500
0.300	385.030100	192.141600
0.310	402.325200	192.098100
0.320	420.294800	192.063800
0.330	439.015700	192.038900
0.340	458.574900	192.023400
0.350	479.072600	192.017300
0.360	500.623600	192.020700
0.370	523.360800	192.033600
0.380	547.438600	192.056000
0.390	573.037800	192.087900
0.400	600.371300	192.129500
0.410	629.691400	192.180500
0.420	661.300100	192.241100
0.430	695.561100	192.311300
0.440	732.917200	192.391100
0.450	773.913100	192.480300
0.460	819.226800	192.578900
0.470	869.711700	192.686900
9.480	926.460300	192.804100
0.490	990.891200	192.930300
0.500	1064.885000	193.065300

Table-4.4 Reciprocal moments of inertia of ^{32}S as functions of β (A = 32 Z = 16 N = 16 $\hbar\omega_0^0 = 10.908$)

β	$\frac{\hbar^2}{2\mathfrak{I}_{crank}}$	$\frac{\hbar^2}{2\mathfrak{I}_{rigid}}$
-0.500	548.020300	193.265300
-0.490	538.614000	192.287800
-0.480	529.116500	191.338600
-0.470	519.527600	190.416700
-0.460	509.847100	189.521000
-0.450	500.075600	188.650300
-0.440	490.213300	187.803700
-0.430	480.261500	186.980300
-0.420	470.221200	186.179100
-0.410	460.093800	185.399300
-0.400	449.881400	184.640300
-0.390	439.585700	183.901200
-0.380	429.208600	183.181400
-0.370	418.752900	182.480200
-0.360	408.220500	181.797100
-0.350	397.614400	181.131500
-0.340	386.937000	180.482700
-0.330	376.191400	179.850400
-0.320	365.380300	179.234000
-0.310	354.506700	178.633100
-0.300	343.573400	178.047200
-0.290	332.583700	177.476000
-0.280	321.540300	176.919000
-0.270	310.446500	176.375900
-0.260	299.305000	175.846200
-0.250	288.119000	175.329900
-0.240	276.891300	174.826300
-0.230	265.624600	174.335300
-0.220	254.321800	173.856600
-0.210	242.985500	173.389900
-0.200	231.618100	172.934900

Table-4.4 (Continued)

-0.190	220.222300	172.491400
-0.180	208.799800	172.059200
-0.170	197.353200	171.638000
-0.160	185.884000	171.227600
-0.150	174.394300	170.827900
-0.140	162.885500	170.438600
-0.130	151.358800	170.059400
-0.120	139.815400	169.690400
-0.110	128.255900	169.331300
-0.100	116.681200	168.981900
-0.090	105.091300	168.642100
-0.080	93.486510	168.311700
-0.070	81.866290	167.990600
-0.060	70.230280	167.678800
-0.050	58.577240	167.375900
-0.040	46.906120	167.082000
-0.030	35.215160	166.797000
-0.020	23.502260	166.520600
-0.010	11.764880	166.252900
0.000	.000261	165.993700
0.010	11.795180	165.743000
0.020	23.625170	165.500600
0.030	35.494220	165.266600
0.040	47.407260	165.040800
0.050	59.369950	164.823100
0.060	71.388450	164.613500
0.070	83.469570	164.412000
0.080	95.621090	164.218400
0.090	107.851400	164.032800
0.100	120.169700	163.855100
0.110	132.586300	163.685300
0.120	145.112600	163.523300
0.130	157.760600	163.369100
0.140	170.544300	163.222700
0.150	183.478300	163.084000

Table-4.4 (Continued)

0.160	196.578900	162.953100
0.170	209.864500	162.829900
0.180	223.354700	162.714400
0.190	237.071400	162.606700
0.200	251.038700	162.506600
0.210	265.283100	162.414300
0.220	279.834300	162.329600
0.230	294.725000	162.252700
0.240	309.991400	162.183600
0.250	325.674200	162.122300
0.260	341.818600	162.068600
0.270	358.475700	162.022800
0.280	375.702200	161.984800
0.290	393.563200	161.954700
0.300	412.131100	161.932400
0.310	431.489500	161.918000
0.320	451.733200	161.911600
0.330	472.971500	161.913200
0.340	495.330000	161.922800
0.350	518.954300	161.940500
0.360	544.014800	161.966200
0.370	570.711900	162.000000
0.380	599.281300	162.042000
0.390	630.004700	162.092200
0.400	663.220800	162.150600
0.410	699.339400	162.217200
0.420	738.862500	162.292000
0.430	782.411600	162.375100
0.440	830.765300	162.466500
0.450	884.913100	162.566000
0.460	946.131000	162.673700
0.470	1016.096000	162.789500
0.480	1097.054000	162.913300
0.490	1192.087000	163.045000
0.500	1305.541000	163.184400

Table-4.5 Reciprocal moments of inertia of ^{36}S as functions of β

$$(A = 36 \quad Z = 18 \quad N = 18 \quad \hbar\omega_0^0 = 10.622)$$

β	$\frac{\hbar^2}{2\mathfrak{I}_{crank}}$	$\frac{\hbar^2}{2\mathfrak{I}_{rigid}}$
-0.500	544.238000	164.409900
-0.490	535.632800	163.596600
-0.480	526.897800	162.807400
-0.470	518.031900	162.041100
-0.460	509.035600	161.296800
-0.450	499.908800	160.573600
-0.440	490.652600	159.870700
-0.430	481.268200	159.187200
-0.420	471.757000	158.522500
-0.410	462.120700	157.875800
-0.400	452.361800	157.246600
-0.390	442.482500	156.634200
-0.380	432.485300	156.037900
-0.370	422.373500	155.457400
-0.360	412.150300	154.892200
-0.350	401.818600	154.341500
-0.340	391.382200	153.805100
-0.330	380.844700	153.282600
-0.320	370.210100	152.773600
-0.310	359.482000	152.277500
-0.300	348.664400	151.794100
-0.290	337.761400	151.323100
-0.280	326.777300	150.864000
-0.270	315.715900	150.416700
-0.260	304.581400	149.980700
-0.250	293.377900	149.555800
-0.240	282.109600	149.141900
-0.230	270.780300	148.738500
-0.220	259.394000	148.345500
-0.210	247.954300	147.962700
-0.200	236.465100	147.589700

Table-4.5 (Continued)

-0.190	224.929600	147.226400
-0.180	213.351700	146.872700
-0.170	201.734100	146.528300
-0.160	190.079900	146.193000
-0.150	178.392000	145.866800
-0.140	166.672800	145.549300
-0.130	154.924600	145.240500
-0.120	143.149300	144.940200
-0.110	131.348700	144.648300
-0.100	119.524000	144.364600
-0.090	107.676500	144.089100
-0.080	95.806760	143.821600
-0.070	83.915240	143.562000
-0.060	72.001590	143.310100
-0.050	60.065510	143.065900
-0.040	48.106110	142.829300
-0.030	36.121970	142.600300
-0.020	24.111300	142.378600
-0.010	12.071650	142.164300
0.000	.000271	141.957300
0.010	12.106490	141.757400
0.020	24.252580	141.564700
0.030	36.442800	141.379000
0.040	48.682530	141.200300
0.050	60.977860	141.028600
0.060	73.335590	140.863900
0.070	85.763710	140.706000
0.080	98.270660	140.554800
0.090	110.866100	140.410600
0.100	123.560800	140.273000
0.110	136.366500	140.142200
0.120	149.296300	140.018100
0.130	162.364700	139.900700
0.140	175.587500	139.789900
0.150	188.982400	139.685900

Table-4.5 (Continued)

0.160	202.569000	139.588400
0.170	216.368600	139.497600
0.180	230.405000	139.413400
0.190	244.704700	139.335900
0.200	259.297000	139.265000
0.210	274.214100	139.200800
0.220	289.492300	139.143200
0.230	305.171900	139.092200
0.240	321.298400	139.048000
0.250	337.922300	139.010500
0.260	355.100400	138.979600
0.270	372.897600	138.955600
0.280	391.386700	138.938200
0.290	410.651300	138.927700
0.300	430.786100	138.924000
0.310	451.900700	138.927200
0.320	474.121300	138.937300
0.330	497.594200	138.954300
0.340	522.490700	138.978300
0.350	549.011800	139.009400
0.360	577.395400	139.047500
0.370	607.925700	139.092600
0.380	640.943400	139.144900
0.390	676.863400	139.204300
0.400	716.193300	139.270900
0.410	759.563200	139.344700
0.420	807.763000	139.425800
0.430	861.798700	139.514100
0.440	922.971600	139.609700
0.450	992.994500	139.712400
0.460	1074.168000	139.822400
0.470	1169.661000	139.939600
9.480	1283.958000	140.063900
0.490	1423.624000	140.195100
0.500	1598.674000	140.333200

CHAPTER 5

PROJECT-3

Perturbation Treatment for the Vibrations of Membranes Subjected to A Restorative Force

5.1 Introduction

Most of the physically important partial differential equations are of second order and can be classified into three types: elliptic, parabolic, and hyperbolic. Elliptic partial differential equations involve second order derivatives in each of the independent variables, each derivative having the same sign when all terms in the equation are grouped on one side. Roughly speaking, parabolic partial differential equations involve only a first-order derivative in one variable but have second order derivatives in the remaining variables. Hyperbolic partial differential equations involve second-order derivatives of opposite sign, such as the wave equation describing the vibrations of a stretched string. Hyperbolic partial differential equations are very essential in engineering and theoretical physics problems. One of the famous problems of their applicability in theoretical physics is the solution of the motion of a relativistic quantum mechanical particle in an electromagnetic field. The problems of vibrating rectangular or circular membrane are also very interesting especially when the membrane is subjected to a restorative force.

The vibrating membrane problem can be used as a rather appropriate example to demonstrate the power of computer algebra systems like Axiom, Maple, Mathematica, Derive, etc. [119]. Different methods have been applied for the investigation of vibrating membranes. The differential quadrature method was applied for frequency analysis of rectangular and circular membranes [46,47]. Accordingly, some important studies concerning analysis of membranes have been carried out [40,41]. Furthermore, free vibration analysis of plates and shells has been also investigated [61,62].

Once the boundary and initial conditions are given, the simplest method for solving the differential equation governing the problem of a vibrating rectangular or circular membrane is given, as usual [114], by separating the variables. In the presence of a restorative force, that is proportional to the velocity, the perturbation expansions for eigenvalues and eigenfunctions are also of particular interest. Based upon the known solutions of the problem in the absence of the restorative force one can then derive the solutions of the problem in the presence of the external force in the form of a power series of those solutions [120].

In the present project, we have solved the differential equation, which represents the motion of a stretched elastic rectangular or circular membrane which is subjected to a restorative force proportional to the velocity by using two methods. The first is the usual method of separation of variables and the second is the perturbation method. The roles of the restoring force and the initial displacement are discussed. In the first part of this chapter

we investigate the circular membrane while in the second part we investigate the rectangular membrane. Accordingly, the displacement of the membrane at any given point (x, y) or (r, θ) and instant of time t is given and the numerical solutions are then given by using the program Mathematica.

5.2 The Circular Membrane

5.2-1 Formulation of the Problem

The formulations of the problem in both Cartesian and the polar coordinates are very similar and we see that the presentation of one of which is sufficient. So, we present the case of the vibrating circular membrane, and we will give the final results in each case. The vibrations of a circular membrane when the membrane is subjected to a restorative force proportional to the velocity at any instant of time t is governed by the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} + K \frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right); u = u(r, \theta, t), K = \text{constant}. \quad (5.1)$$

subjected two the boundary conditions

$$u(a, \theta, t) = 0, \quad (5.2)$$

and the initial conditions

$$u(r, \theta, 0) = f(r, \theta), \quad (5.3)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(r, \theta). \quad (5.4)$$

The initial displacement $f(r, \theta)$ and initial velocity $g(r, \theta)$ of the membrane are assumed to be continuous functions.

5.2-2 The Method of Separation of Variables

Let us find solutions of (5.1) subject to the boundary and initial conditions given by (5.2), (5.3) and (5.4) in the form

$$u = u(r, \theta, t) = T(t)R(r)\Theta(\theta).$$

Hence, the partial differential equation (5.1) separates to three second-order ordinary differential equations in the form

$$\frac{d^2 T}{dt^2} + k \frac{dT}{dt} = -c^2 \lambda^2 T \quad (5.5)$$

$$\frac{d^2 \Theta}{d\theta^2} = -m^2 \Theta, \quad (5.6)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\lambda^2 - \frac{m^2}{r^2} \right) R = 0. \quad (5.7)$$

In equations (5.5), (5.6) and (5.7) m and λ are arbitrary constants to be determined. The solutions of these equations, which are finite in t and continuous in our domain of values of r and θ are simply given by

$$T(t) = e^{\frac{-kt}{2}} \{A_1 \cos(c\beta t) + B_1 \sin(c\beta t)\}, \quad (5.8)$$

$$\Theta(\theta) = A_2 \cos(m\theta) + B_2 \sin(m\theta), \quad (5.9)$$

$$R(r) = J_m(\lambda r), \quad (5.10)$$

Here, $m = 0, 1, 2, \dots$, $\beta = \sqrt{\lambda^2 - \frac{K^2}{4c^2}}$ and $J_m(\lambda r)$ are the Bessel functions. The boundary condition (5.2) implies that

$$R(a) = J_m(\lambda a) = 0, \quad (5.11)$$

Assuming that $j_{m,1}, j_{m,2}, j_{m,3}, \dots, j_{m,n}, \dots$, are positive roots of $J_m(r)$, the solutions of (5.1) are then given by

$$u = e^{\frac{-kt}{2}} J_m\left(\frac{j_{m,n}}{a} r\right) \{A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)\} \{\cos(c\beta_{m,n} t) + D_{m,n} \sin(c\beta_{m,n} t)\}, \quad (5.12)$$

where

$$\beta_{m,n} = \sqrt{\frac{j_{m,n}^2}{a^2} - \frac{K^2}{4c^2}}.$$

If the function $g(r, \theta)$ is taken to be zero, i.e. the initial velocity is zero, we obtain

$D_{m,n} = \frac{K}{2c\beta_{m,n}}$, and the final solution in this case is then given by

$$u = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} e^{\frac{-kt}{2}} \left\{ \cos(c\beta_{m,n} t) + \frac{K}{2c\beta_{m,n}} \sin(c\beta_{m,n} t) \right\} J_m\left(\frac{j_{m,n}}{a} r\right) \{A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)\} \quad (5.13)$$

5.2-3 The Perturbation Method of Solution

To apply the perturbation method [120,121] to equation (5.1) we try first to find solutions of the form

$$u = v e^{\frac{-kt}{2}} \quad (5.14)$$

where v is also a function of r, θ and t .

Equation (5.1), then, becomes

$$\frac{\partial^2 v}{\partial t^2} - \frac{K^2}{4} v = c^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right), \quad (5.15)$$

with the boundary condition

$$v(a, \theta, t) = 0, \quad (5.16)$$

and the initial conditions

$$v(r, \theta, 0) = f(r, \theta), \quad (5.17)$$

$$v_{t=0} = g(r, \theta) + \frac{K}{2} f(r, \theta) = h(r, \theta), \quad (5.18)$$

The order of the approximation in the perturbation method is very important, so that we rewrite equation (5.15) in the form

$$c^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right) = v_{tt} - \alpha \frac{K^2}{4} v, \quad (5.19)$$

where α is a parameter introduced to know the order of the approximation and takes the value 1 in the final result. Accordingly, the zeroth order of the approximation is governed by the equation representing the case where there is no external restorative force, which is obtained by putting $\alpha = 0$ in (5.19), i.e.

$$c^2 \left(\frac{\partial^2 v^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial v^{(0)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v^{(0)}}{\partial \theta^2} \right) = v_{tt}^{(0)}, \quad (5.20)$$

where $v^{(0)}$ is the corresponding solution in this case, which can be proved to be given by

$$v^{(0)}(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{j_{m,n}}{a} r \right) \{ A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta) \} \\ \times \{ \cos(cj_{m,n}t) + D_{m,n} \sin(cj_{m,n}t) \}. \quad (5.21)$$

The boundary condition (5.16) is then satisfied and the initial conditions are the same as given by (5.17) and (5.18), with $k = 0$.

We have now to determine by how much the solutions of the vibrations of the circular membrane, given by equation (5.21), under the boundary and initial conditions stated above, have been changed on account of the presence of the disturbing factor $K \frac{\partial u}{\partial t}$, since it is assumed to be small compared to the other terms. The change is known as a perturbation.

The second step is now to use the solution $v^{(0)}(r, \theta, t)$, equation (5.21), to derive solutions of equation (5.19), satisfying the boundary and initial conditions stated above in the following manner. Assume that the solutions of equations (5.19), $v(r, \theta, t)$, are expanded in series in powers of α , such that

$$v = v^{(0)} + \alpha v^{(1)} + \alpha^2 v^{(2)} + \dots \quad (5.22)$$

For any order of the approximation, we have the system of inhomogeneous wave equations

$$c^2 \left(\frac{\partial^2 v^{(j)}}{\partial r^2} + \frac{1}{r} \frac{\partial v^{(j)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v^{(j)}}{\partial \theta^2} \right) = v_{tt}^{(j)} - \frac{K^2}{4} v^{(j-1)}, \quad j = 1, 2, 3, \dots, \quad (5.23)$$

with the initial conditions

$$v^{(j)}(r, \theta, 0) = 0, \quad v_{t=0}^{(j)} = 0, \quad j = 1, 2, 3, \dots \quad (5.24)$$

The solutions of equations (5.23) are given by

$$v^{(j)}(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \mathcal{W}_{m,n}^{(j)}(t) J_m \left(\frac{j_{m,n}}{a} r \right) \{A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)\}, \quad (5.25)$$

$$j = 0, 1, 2, \dots,$$

where

$$\mathcal{W}_{m,n}^{(0)}(t) = \cos(cj_{m,n}t) + D_{m,n} \sin(cj_{m,n}t), \quad (5.26)$$

and $\mathcal{W}_{m,n}^{(j)}(t)$, $j = 1, 2, 3, \dots$, are to be determined. From (5.23) and (5.25) we get

$$\frac{d^2 \mathcal{W}_{m,n}^{(j)}(t)}{dt^2} + (cj_{m,n})^2 \mathcal{W}_{m,n}^{(j)}(t) = \frac{K^2}{4} \mathcal{W}_{m,n}^{(j-1)}(t), \quad j = 1, 2, 3, \dots \quad (5.27)$$

The solutions of (5.27) can be written in the form

$$\mathcal{W}_{m,n}^{(j)}(t) = \frac{K^2}{4cj_{m,n}} \int_0^t \mathcal{W}_{m,n}^{(j-1)}(\tau) \sin\{cj_{m,n}(t-\tau)\} d\tau, \quad j = 1, 2, 3, \dots \quad (5.28)$$

Hence, the solutions of (5.1) are finally given by

$$\begin{aligned} u(r, \theta, t) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u_{m,n}(r, \theta, t) \\ &= e^{-\frac{Kt}{2}} \sum_{j=0}^{\infty} \alpha^j \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \mathcal{W}_{m,n}^{(j)}(t) J_m \left(\frac{j_{m,n}}{a} r \right) \{A_{m,n} \cos(m\theta) + \\ &\quad + B_{m,n} \sin(m\theta)\}. \end{aligned} \quad (5.29)$$

From (5.27) and (5.28), we have

$$\mathcal{W}_{m,n}^{(1)}(t) = \frac{K^2}{8cj_{m,n}} \left[E_{m,n} t \sin(cj_{m,n}t) + L_{m,n} \left(\frac{\sin(cj_{m,n}t)}{cj_{m,n}} - t \cos(cj_{m,n}t) \right) \right], \quad (5.30)$$

and

$$\begin{aligned} \mathcal{W}_{m,n}^{(2)}(t) = & \frac{K^4}{128(c_{j_{m,n}})^2} \left[M_{m,n} \left\{ \frac{t \sin(c_{j_{m,n}}t)}{c_{j_{m,n}}} - t^2 \cos(c_{j_{m,n}}t) \right\} + \right. \\ & N_{m,n} \left\{ \frac{3}{(c_{j_{m,n}})^2} \sin(c_{j_{m,n}}t) - \frac{3t}{c_{j_{m,n}}} \cos(c_{j_{m,n}}t) - t^2 \sin(c_{j_{m,n}}t) + \right. \\ & \left. \left. \frac{3}{(c_{j_{m,n}})^2} \left\{ \sin(c_{j_{m,n}}t) \right\}^3 - \frac{6 \sin^2(c_{j_{m,n}}t) \cos(c_{j_{m,n}}t)}{c_{j_{m,n}}} \right\} \right]. \end{aligned} \quad (5.31)$$

Accordingly, $v^{(1)}(r, \theta, t)$ and $v^{(2)}(r, \theta, t)$ are calculated. From which the functions v and, hence, u are calculated. We have applied the perturbation method to the second order of the approximation, which is sufficiently enough since the convergence of the solutions for our choice of parameters is good. Hence, our solutions are finally given by

$$u(r, \theta, t) = e^{-\frac{Kt}{2}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{j_{m,n}r}{a} \right) [A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)] \mathcal{W}_{m,n}(t), \quad (5.32)$$

where

$$\begin{aligned} \mathcal{W}_{m,n}(t) = & \left\{ \cos(c_{j_{m,n}}t) + D_{m,n} \sin(c_{j_{m,n}}t) \right\} + \frac{K^2}{8c_{j_{m,n}}} \left\{ E_{m,n} t \sin(c_{j_{m,n}}t) + \right. \\ & L_{m,n} \left(\frac{\sin(c_{j_{m,n}}t)}{c_{j_{m,n}}} - t \cos(c_{j_{m,n}}t) \right) \left. \right\} + \frac{K^4}{128(c_{j_{m,n}})^2} \left[M_{m,n} \left\{ \frac{t \sin(c_{j_{m,n}}t)}{c_{j_{m,n}}} - t^2 \cos(c_{j_{m,n}}t) \right\} + \right. \\ & N_{m,n} \left\{ \frac{3}{(c_{j_{m,n}})^2} \sin(c_{j_{m,n}}t) - \frac{3t}{c_{j_{m,n}}} \cos(c_{j_{m,n}}t) - t^2 \sin(c_{j_{m,n}}t) + \right. \\ & \left. \left. \frac{3}{(c_{j_{m,n}})^2} \left\{ \sin(c_{j_{m,n}}t) \right\}^3 - \frac{6 \sin^2(c_{j_{m,n}}t) \cos(c_{j_{m,n}}t)}{c_{j_{m,n}}} \right\} \right] \end{aligned} \quad (5.33)$$

The initial condition (5.3) now gives

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{j_{m,n}r}{a} \right) \{A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)\} = f(r, \theta), \quad (5.34)$$

and the initial condition (5.4) gives

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{j_{m,n}r}{a} \right) \{A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)\} \left(\frac{-K}{2} + c\lambda_{m,n} D_{m,n} \right) = g(r, \theta), \quad (5.35)$$

5.2-4 Determination of the Coefficients

In the treatment of the present problem we considered the following physical assumptions:

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the plane.
3. The tension per unit length, T , caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.

4. The deflection $u(r, \theta, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Although these assumptions cannot be realized exactly, they hold relatively accurately for small transverse vibrations of a thin elastic membrane, so that we shall obtain a good model, for instance, of a drumhead. In the numerical calculations we take $a = 1$ ft, the density, $\rho = 2$ slugs/ft², as for light rubber, the constant tension $T = 8$ lb/ft, so that $c^2 = T/\rho = 4(\text{ft}/\text{sec})^2$. Moreover, the initial displacement is taken to be

$$f(r, \theta) = \sum_{m=0}^{\infty} 2^{-m} J_m(j_{m,1}r) \cos(m\theta), \quad (5.36)$$

which is a continuous function of r and θ in the intervals $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

Also, as before, we assume that the initial velocity is zero, so that $g(r, \theta) = 0$. Hence,

$$D_{m,n} = \frac{K}{2cj_{m,n}} \quad (5.37)$$

The initial displacement condition then gives

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(j_{m,n}r) \{A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)\} = \sum_{m=0}^{\infty} 2^{-m} J_m(j_{m,1}r) \cos(m\theta). \quad (5.38)$$

Comparing the coefficients of $\cos(m\theta)$ and $\sin(m\theta)$ on both sides of (5.38) we conclude that all the coefficients are zero except:

$$A_{m,1} = 2^{-m}, \quad m = 0, 1, 2, 3, \dots \dots \dots \quad (5.39)$$

Accordingly, (5.32) takes the form

$$u(r, \theta, t) = e^{-\frac{Kt}{2}} \sum_{m=0}^{\infty} J_m(j_{m,1}r) 2^{-m} \cos(m\theta) \mathcal{W}_{m,1}(t) \quad (5.40)$$

where $\mathcal{W}_{m,1}(t)$ are given by (5.33) and (5.37).

It is to be noticed that the boundary and initial conditions for our model do not give information about the coefficients $E_{m,1}$, $L_{m,1}$, $M_{m,1}$ and $N_{m,1}$. We can arbitrarily take $L_{m,1} = 0$ and $N_{m,1} = 0$. Also, choosing $E_{m,1} = 1$ and $M_{m,1} = 1$ we get

$$\begin{aligned} W_{m,1}(t) = & \cos(cj_{m,1}t) + \frac{K}{2cj_{m,1}} \sin(cj_{m,1}t) + \frac{K^2}{8cj_{m,1}} \{t \sin(cj_{m,1}t)\} + \\ & + \frac{K^4}{128(cj_{m,1})^2} \left[\frac{t \sin(cj_{m,n}t)}{cj_{m,n}} - t^2 \cos(cj_{m,n}t) \right] \end{aligned} \quad (5.41)$$

5.2-5 Results and Conclusions for the Circular Membrane

In order to investigate the role of the restorative force, given in this thesis by the term $K \frac{\partial u}{\partial t}$, we draw the maximum value of the function u as function of the time t for different values of the coefficient K . In Fig. (1) we present the variation of the maximum value of u with respect to the time t when there is no external force, $K = 0$. Fig. (2) shows the variations of maximum u with respect to t for $K = 0.1, 0.2, 0.3, 0.4$ and 0.5 . It is seen from Fig.(2)

that the amplitude of the oscillation and the time during which the phenomena is seen, decrease as K increases, as expected. The process is well presented at $K = 0.3$.

In Fig. (3) we present the vibrations of the circular membrane at $t = 0$, $K = 0.3$, initial displacement. In Figs. (4-21 "a") we present the vibrations of the circular membrane, by using the method of separation of variables, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \dots, 16, 27, 28$, respectively, $K = 0.3$. In Figs. (4-21 "b") we present the vibrations of the circular membrane, by using the perturbation method, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \dots, 16, 27, 28$, respectively, $K = 0.3$.

It is seen from Figs. (4-21 "a, b") that the two methods of solutions give the same results (same vibrations), a result which shows that the perturbation calculations up to the second order of the approximation give accurate solution to the original problem.

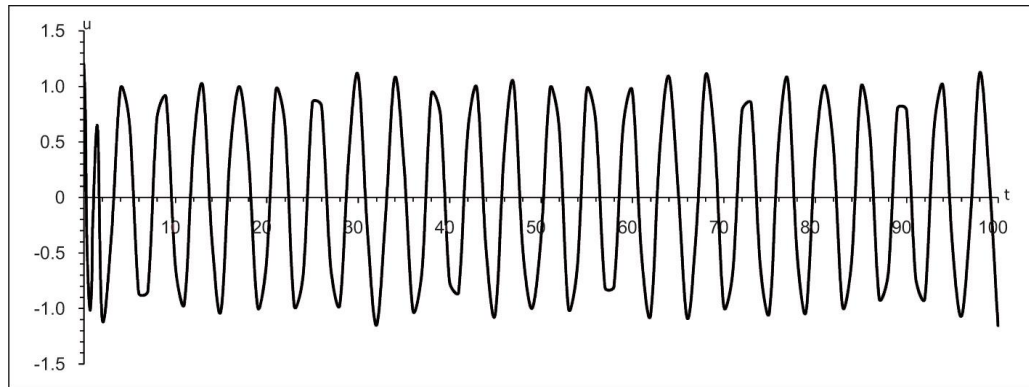


Fig. (1) Maximum displacement, for different values of r, θ , as function of time t , $K = 0.0$

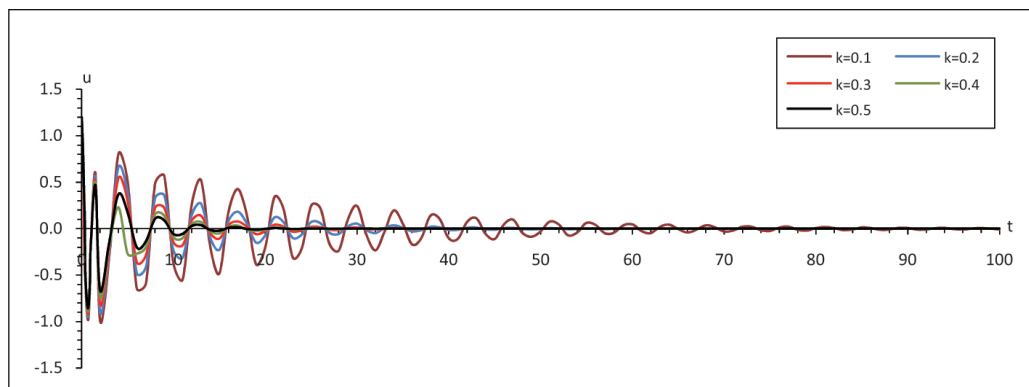


Fig. (2) Maximum displacement, for different values of r, θ , as function of time t , $K = 0.1, 0.2, 0.3, 0.4$ and 0.5 .

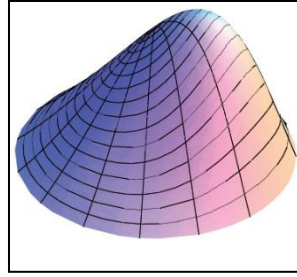
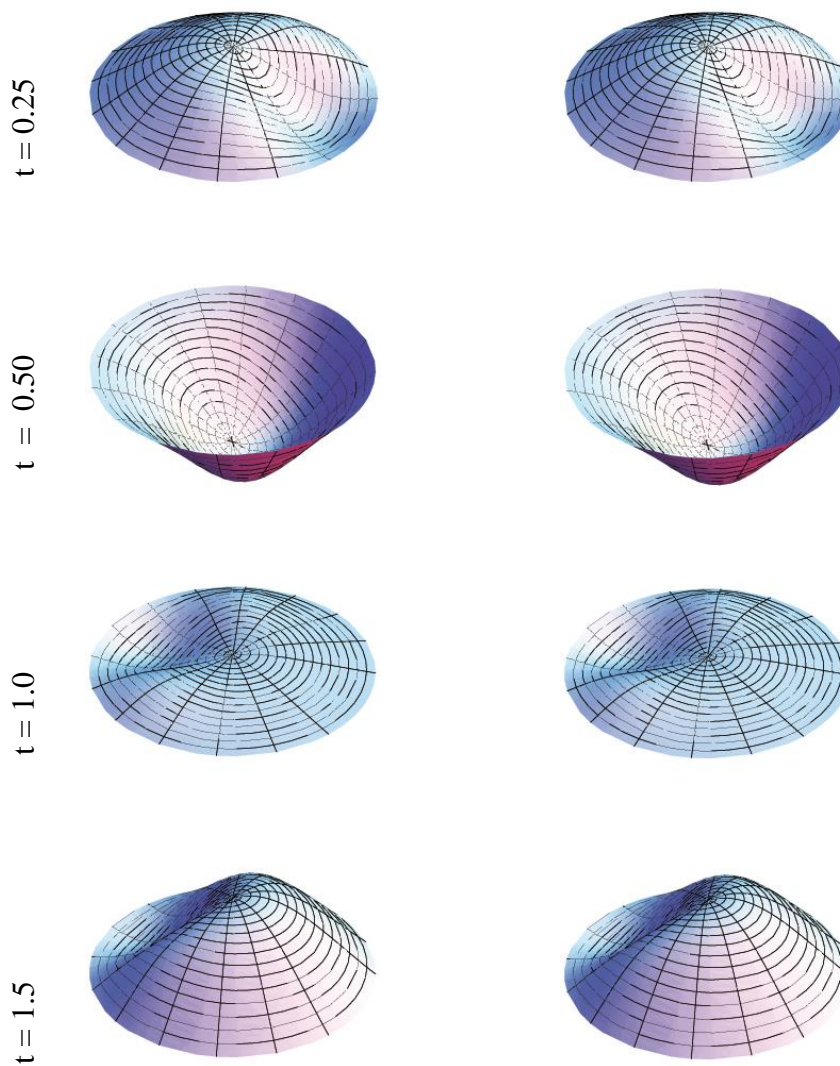
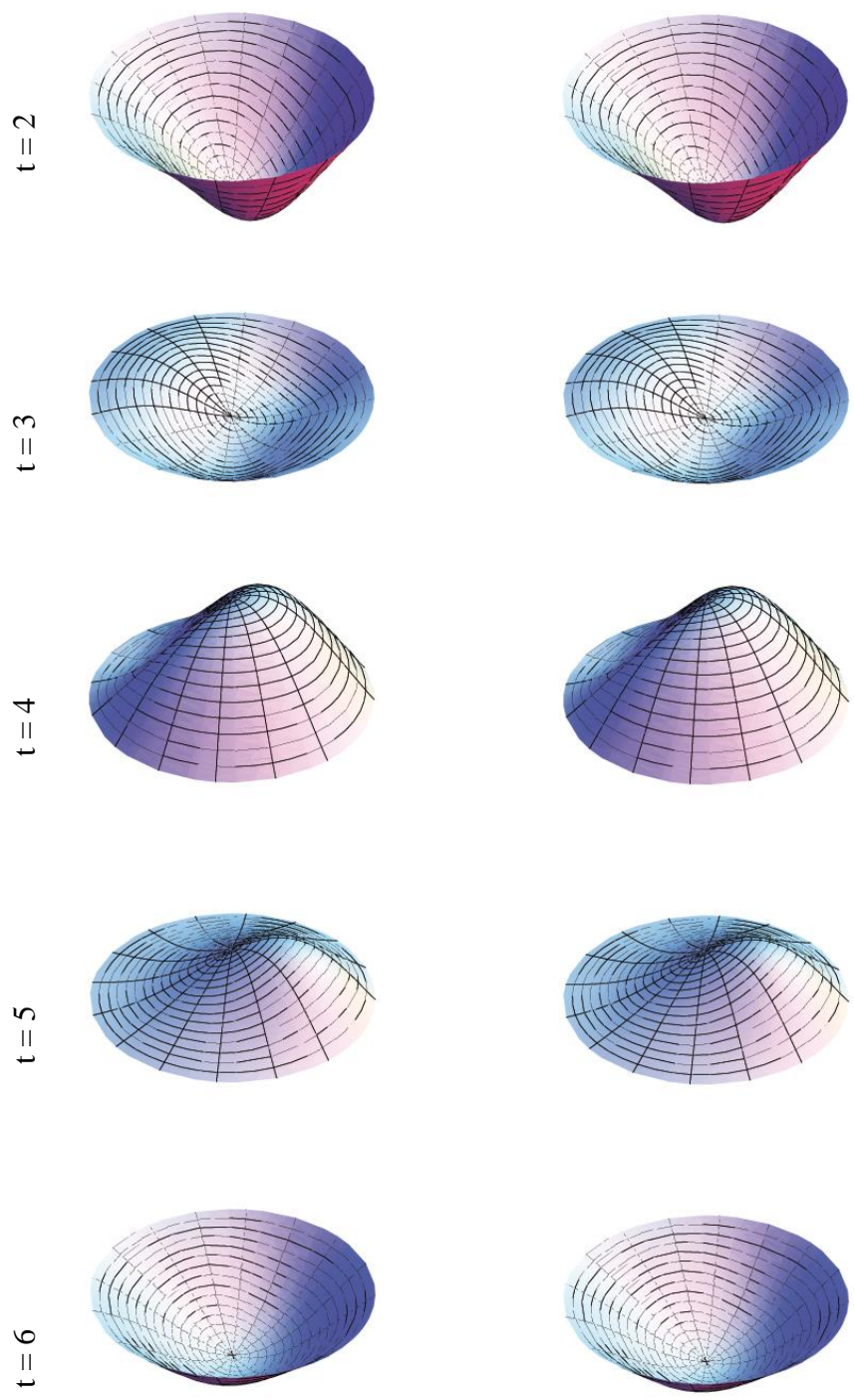
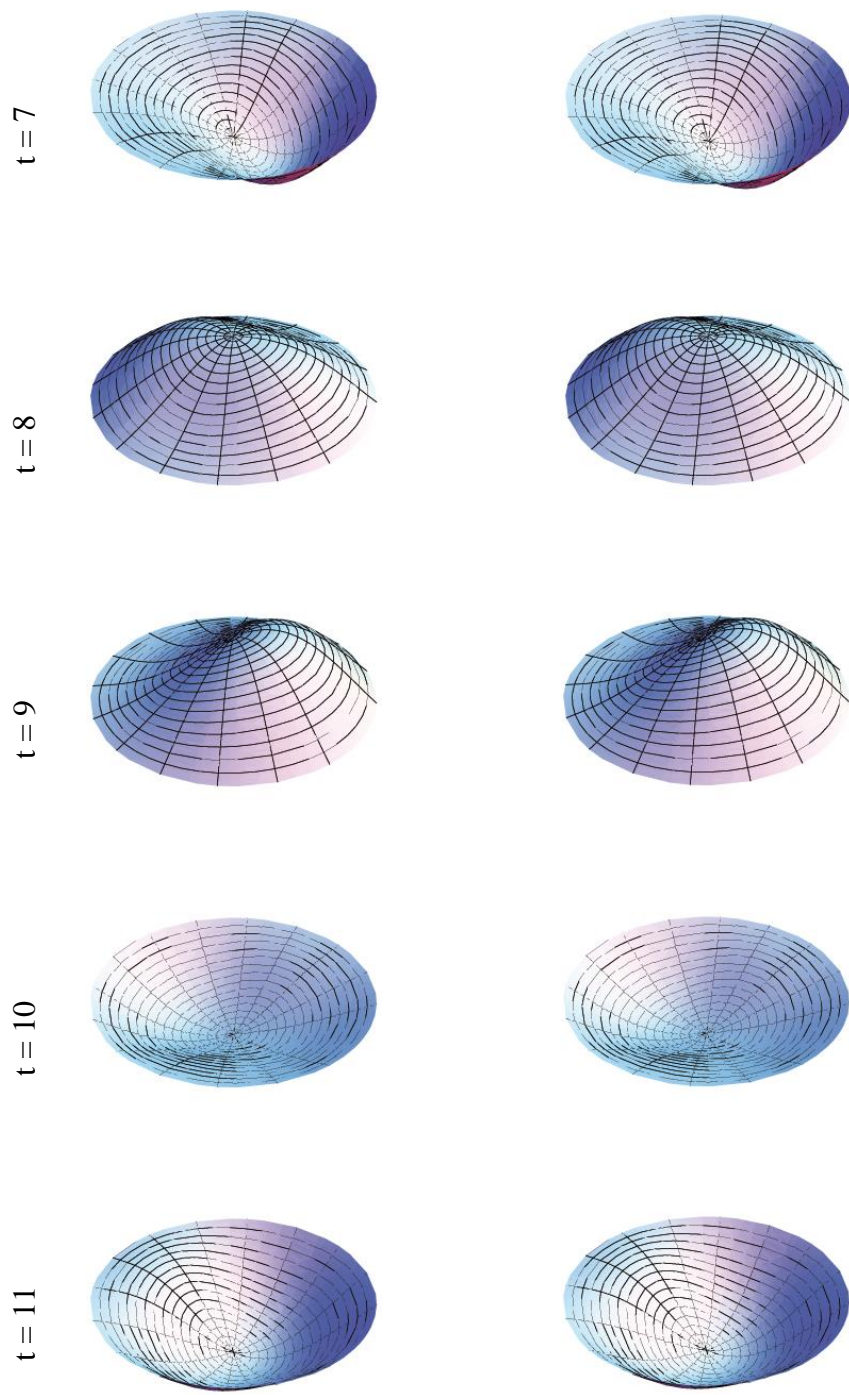


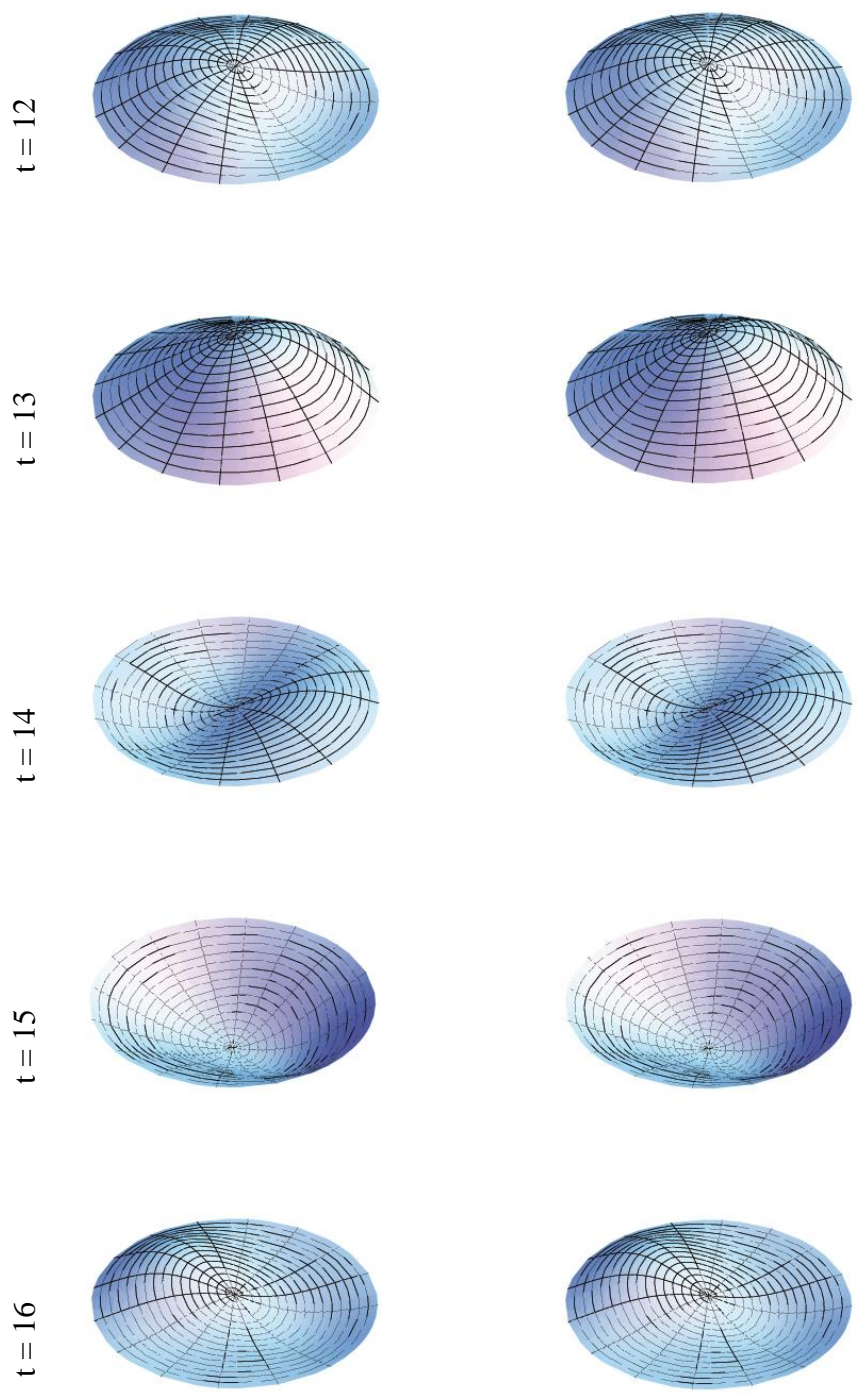
Fig. (3) Initial displacement.

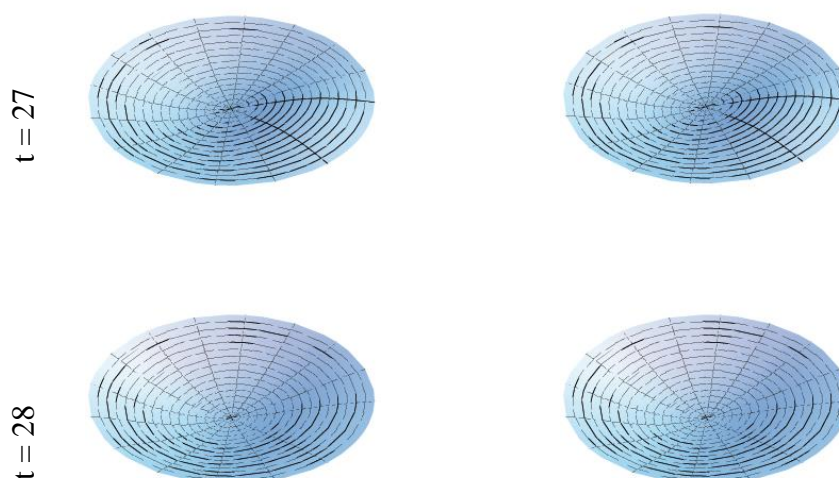
Sec. a- Method of separation of variables b- Second-order perturbation method











Figs. (4-21 "a") Vibrations of the circular membrane, by using the method of separation of variables, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \dots, 16, 27, 28$, respectively, $K = 0.3$.

Figs. (4-21 "b") Vibrations of the circular membrane, by using the perturbation method, at $t = 0.25, 0.5, 1, 1.5, 2, 3, \dots, 16, 27, 28$, respectively, $K = 0.3$.

Since the radius of the membrane $a = 1$ ft, the values of the frequency of oscillation, in our model, are given by

$$f_{m,1} = \frac{\lambda_{m,1}c}{2\pi} \text{ HZ.} \quad (5.42)$$

The values of these frequencies are given in Table-1.

Table-1 Frequency values for $c = 2$ ft/sec.

	Frequency (HZ)
f_{01}	0.765
f_{11}	1.219
f_{21}	1.634
f_{31}	2.030
f_{41}	2.414
f_{51}	2.791

f_{61}	3.161
f_{71}	3.527
f_{81}	3.890
f_{91}	4.250

In practical applications, the frequencies given by (5.42) produce vibrations which are outside the range of hearing. In circumstances like what we have studied, it is desirable from the physical point of view to take the value of the velocity as $c = 800$ ft/sec. In such case, the corresponding values of the frequency are in the range $306.052 \leq f_{m1} \leq 1699.523$ HZ, $m = 0, 1, \dots, 9$. For the considered modes, these frequencies produce vibrations which can be heard. At the same time, the shape of the resulting figures are very similar to those obtained in Figures (4-21 “a” and “b”) but with different amplitudes.

5.3 The Rectangular Membrane

5.3-1 The Differential Equation

The differential equation which governs the motion of the vibrating rectangular membrane is given by [8]

$$\frac{\partial^2 u_0}{\partial t^2} = c^2 \left\{ \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \right\}, \quad (5.43)$$

where c is a constant, which is given in terms of the tension per unit lengths and the density ρ by the relation

$$c = \sqrt{\frac{T}{\rho}}, \text{ ft/sec.} \quad (5.44)$$

The constant c has the dimension of velocity, as expected. The solutions of equation (5.43) are well known when the boundary and initial conditions are stated [8].

When the membrane is subjected to a restorative force that is proportional to the velocity the new differential equation is now given by

$$\frac{\partial^2 u}{\partial t^2} + k \frac{\partial u}{\partial t} = c^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\}, \quad (5.45)$$

where k is the proportionality constant.

5.3-2 The Boundary Conditions

We take u to be zero on the boundary of the membrane, as for all applications, so that

$$u = 0, \quad 0 \leq x \leq a \text{ and } 0 \leq y \leq b, \quad (5.46)$$

for all $t \geq 0$.

5.3-3 The Initial Conditions

The initial displacement is taken, as in practical applications, as follows

$$u(x, y, 0) = f(x, y), \quad (5.47)$$

where $f(x, y)$ is assumed to be a continuous function.

The initial velocity is

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x, y), \quad (5.48)$$

where $g(x, y)$ is also a continuous function.

5.3-4 The Method of Separation of Variables

By separating the variables, one can easily obtain the following solutions for equation (5.43) under the stated above boundary and initial conditions:

$$u_0(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\left\{ A_{m,n} \cos(\lambda_{m,n} t) + B_{m,n} \sin(\lambda_{m,n} t) \right\} \times \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \right], \quad (5.49)$$

where

$$A_{m,n} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy, \quad (5.50)$$

$$B_{m,n} = \frac{4}{ab\lambda_{m,n}} \int_0^a \int_0^b g(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy, \quad (5.51)$$

And

$$\lambda_{m,n} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m = 1, 2, \dots \text{ and } n = 1, 2, \dots \quad (5.52)$$

Furthermore, by separating the variables one obtains the following solutions for equation (5.45)

$$u(x, y, t) = e^{-kt/2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ A_{m,n} \cos(\mu_{m,n} t) + D_{m,n} \sin(\mu_{m,n} t) \right\} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad (5.53)$$

where $A_{m,n}$ is given by (5.50) and

$$D_{m,n} = \frac{4}{ab\mu_{m,n}} \int_0^a \int_0^b \left\{ g(x,y) + \frac{c\pi}{2} f(x,y) \right\} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy. \quad (5.54)$$

In equations (5.53) and (5.54) the values $\mu_{m,n}$ are given by

$$\mu_{m,n} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{k^2}{4c^2\pi^2}}, \quad (5.55)$$

where $m = 1, 2, \dots$ and $n = 1, 2, \dots$.

The values $\mu_{m,n}$ are called eigenvalues of the rectangular membrane and the functions

$$u_{m,n}(x, y, t) = e^{-kt/2} \left\{ A_{m,n} \cos(\mu_{m,n} t) + D_{m,n} \sin(\mu_{m,n} t) \right\} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad (5.56)$$

are called eigenfunctions of the rectangular membrane.

5.3-5 The Perturbation Method of Solution

We have now to determine by how much the solutions of equation (5.43), under the boundary and initial conditions stated above, have been changed on account of the presence of the disturbing factor $k \frac{\partial u}{\partial t}$, since it is assumed to be small compared to the other terms. The change is known as a perturbation.

To apply the perturbation method [10] to equation (5.45) we try first to find solutions of the form

$$u = v e^{\frac{-kt}{2}}. \quad (5.57)$$

Equation (5.45), then, becomes

$$c^2(v_{xx} + v_{yy}) = v_{tt} - \frac{k^2}{4} v \quad (5.58)$$

with the initial conditions

$$\begin{aligned} v(x, y, 0) &= f(x, y), \\ v_t|_{t=0} &= g(x, y) + \frac{k}{2} f(x, y) = h(x, y) \end{aligned}$$

To apply the perturbation method to equation (5.58) we rewrite it in the form

$$c^2(v_{xx} + v_{yy}) = v_{tt} - \alpha \frac{k^2}{4} v, \quad (5.59)$$

where α is a parameter introduced to know the order of the approximation and takes the value 1 in the final result. Accordingly, the equation representing the case where there is no external restorative force is obtained by putting $\alpha = 0$ in (5.59)

$$c^2(v_{xx}^{(0)} + v_{yy}^{(0)}) = v_{tt}^{(0)}, \quad (5.60)$$

where $v^{(0)}$ is the corresponding solution in this case.

The second step now is to use the solution $v^{(0)}(x, y, t)$ of equation (5.60) to derive solutions of equation (5.59), satisfying the boundary and initial conditions stated above in the following manner. Assume that the solutions of equations (5.59), $v(x, y, t)$, are expanded in series in powers of α , such that

$$v = v^{(0)} + \alpha v^{(1)} + \alpha^2 v^{(2)} + \dots \quad (5.61)$$

Substituting from (5.61) into (5.59) and equating the coefficients of the different powers of α in both sides we get

$$c^2(v_{xx}^{(0)} + v_{yy}^{(0)}) = v_{tt}^{(0)},$$

$$c^2(v_{xx}^{(j)} + v_{yy}^{(j)}) = v_{tt}^{(j)} - \frac{k^2}{4} v^{(j-1)}, \quad j = 1, 2, 3, \dots \quad (5.62)$$

Thus, in the presence of the perturbation we have for the zero's order of the approximation the same equation (5.60), as expected, with the boundary and initial conditions given by

$$v^{(0)}(x, y, 0) = f(x, y), \quad (5.63)$$

$$v^{(0)}_t|_{t=0} = h(x, y). \quad (5.64)$$

For any order of the approximation, we have the system of inhomogeneous wave equations given by (5.62) with the initial conditions

$$v^{(j)}(x, y, 0) = 0, \quad (5.65)$$

$$v^{(j)}_t|_{t=0} = 0; \quad j = 1, 2, 3, \dots \quad (5.66)$$

The solutions of equations (5.60) and (5.62) with the given boundary and initial conditions are given by

$$v^{(j)}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{m,n}^{(j)}(t) \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right), \quad j = 0, 1, 2, \dots, \quad (5.67)$$

where

$$w_{m,n}^{(0)}(t) = A_{m,n} \cos(\lambda_{m,n}t) + E_{m,n} \sin(\lambda_{m,n}t). \quad (5.68)$$

In equation (5.68) $A_{m,n}$ is given as before by (5.50) and

$$E_{m,n} = \frac{4}{abc\lambda_{m,n}} \int_{y=0}^b \int_{x=0}^a h(x,y) \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right) dx dy, \quad (5.69)$$

$$\lambda_{m,n} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}. \quad (5.70)$$

From (5.62) and (5.67) we get

$$\frac{d^2 w_{m,n}^{(j)}(t)}{dt^2} + (\lambda_{m,n})^2 w_{m,n}^{(j)}(t) = \frac{K^2}{4} w_{m,n}^{(j-1)}(t), \quad j = 1, 2, 3, \dots \quad (5.71)$$

The solutions of (5.71) can be written in the form:

$$w_{m,n}^{(j)}(t) = \frac{K^2}{4\lambda_{m,n}} \int_0^t w_{m,n}^{(j-1)}(\tau) \sin\{\lambda_{m,n}(t-\tau)\} d\tau, \quad j = 1, 2, 3, \dots \quad (5.72)$$

Hence, the solutions of (5.45) are finally given by

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{m,n}(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} e^{-\frac{kt}{2}} w_{m,n}^{(j)}(t) \alpha^j \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right). \quad (5.73)$$

From (5.68) and (5.72), we have

$$w_{m,n}^{(1)}(t) = \frac{k^2}{8\lambda_{m,n}} \left[A_{m,n} t \sin(\lambda_{m,n}t) + E_{m,n} \left(\frac{\sin(\lambda_{m,n}t)}{\lambda_{m,n}} - t \cos(\lambda_{m,n}t) \right) \right], \quad (5.74)$$

and

$$\frac{128(\lambda_{m,n})^2}{k^4} \times w_{m,n}^{(2)}(t) = A_{m,n} \left\{ \frac{t \sin(\lambda_{m,n}t)}{\lambda_{m,n}} - t^2 \cos(\lambda_{m,n}t) \right\} + E_{m,n} \left\{ \frac{\frac{3}{(\lambda_{m,n})^2} \sin(\lambda_{m,n}t) - \frac{3t}{\lambda_{m,n}} \cos(\lambda_{m,n}t) - t^2 \sin(\lambda_{m,n}t)}{(\lambda_{m,n})^2} + \frac{3}{(\lambda_{m,n})^2} \{ \sin(\lambda_{m,n}t) \}^3 - \frac{6 \sin^2(\lambda_{m,n}t) \cos(\lambda_{m,n}t)}{\lambda_{m,n}} \right\}. \quad (5.75)$$

Accordingly, $v^{(1)}(x,y,t)$, $v^{(2)}(x,y,t)$, and so on are calculated. From which the functions v and, hence, u are calculated.

5.3-6 Results and Conclusions

In the numerical computations we have considered a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft, the constant tension is 12.5 lb/ft, the density is 2.5 slugs/ft², as for light rubber. Moreover, the constants k and c are given by

$$k = c\pi/10 \quad \text{and} \quad c^2 = \frac{16}{5\pi^2} \text{ft}^2/\text{sec}^2.$$

1. Nodal Lines of the Rectangular Membrane

It is very interesting to notice that, depending on a and b several functions $u_{m,n}$, (5.29) or (5.73), may correspond to the same eigenvalue. Physically this means that there may exist vibrations which have the same frequency but entirely different nodal lines (curves of points on the membrane that do not move). In our model the eigenvalues are given by (5.55). For $a = 4$ and $b = 2$ we get

$$\frac{20\mu_{m,n}}{\pi c} = \sqrt{N} = \sqrt{25m^2 + 100n^2 - 1}, \quad m = 1, 2, 3, \dots \text{ and } n = 1, 2, 3, \dots \quad (5.76)$$

Accordingly, different functions $u_{m,n}$ may correspond to the same value of $\mu_{m,n}$. For example, we have $N = 1999$, for $m = 4$ and $n = 4$, and also for $m = 8$ and $n = 2$. Hence,

$$\mu_{4,4} = \mu_{8,2} = \frac{c\pi}{20} \sqrt{1999}$$

But for $m = 4$ and $n = 4$, the corresponding function is

$$u_{4,4} = e^{-kt/2} \left\{ A_{4,4} \cos \left(\frac{c\pi}{20} \sqrt{1999}t \right) + D_{4,4} \sin \left(\frac{c\pi}{20} \sqrt{1999}t \right) \right\} \sin(\pi x) \sin(2\pi y). \quad (5.77)$$

And for $m = 8$ and $n = 2$, the corresponding function is

$$u_{8,2} = e^{-kt/2} \left\{ A_{8,2} \cos \left(\frac{c\pi}{20} \sqrt{1999}t \right) + D_{8,2} \sin \left(\frac{c\pi}{20} \sqrt{1999}t \right) \right\} \sin(2\pi x) \sin(\pi y). \quad (5.78)$$

These two functions are certainly different and have the nodal lines $x = 1, x = 2, x = 3$, and $y = \frac{1}{2}, y = 1, y = \frac{3}{2}$ in the first case and $x = \frac{1}{2}, x = 1, x = \frac{3}{2}, x = 2, x = \frac{5}{2}, x = 3, x = \frac{7}{2}$ and $y = 1$, in the second case.

Furthermore, we have calculated several values of m and $n \leq 100$, for which the corresponding eigenvalues are the same and belong to different eigenfunctions. In Table-1 we present only one case from each set of values with the same length (≥ 3). The corresponding nodal lines for them can be easily obtained as discussed above.

Table-1 Values of m , n and N which belong to nodal lines

m	n	N	Length
2	7	4999	3
10	5	4999	3
14	1	4999	3
2	8	6499	4
8	7	6499	4
14	4	6499	4
16	1	6499	4
10	35	125999	5
34	31	125999	5
50	25	125999	5
62	17	125999	5
70	5	125999	5
2	18	32599	6
12	17	32599	6
20	15	32599	6
30	10	32599	6
34	6	32599	6
36	1	32599	6
16	51	266599	7
38	48	266599	7
54	44	266599	7
72	37	266599	7
74	36	266599	7

88	27	266599	7
96	19	266599	7
8	33	110599	8
18	32	110599	8
24	31	110599	8
46	24	110599	8
48	23	110599	8
62	12	110599	8
64	9	110599	8
66	4	110599	8

2. Displacement of the Rectangular Membrane

For the initial displacement we have choose several functions which satisfy the boundary conditions, such as

$$f(x, y) = xy(a^2 - x^2)(b^2 - y^2), \quad (5.79)$$

$$f(x, y) = d(ax - x^2)(by - y^2), d = const, \quad (5.80)$$

which are positive continuous function of x and y in the intervals $x \in [0, a]$ and $y \in [0, b]$. For the sake of illustration, we consider the function given by (5.79). Similarly, for the initial velocity we have considered several functions which produce good results such as

$$g(x, y) = 0 \quad (5.81)$$

$$g(x, y) = x + y, \quad (5.82)$$

$$g(x, y) = x - y \quad (5.83)$$

We have applied the perturbation method to the second-order of the approximation, which is sufficiently enough since the convergence of the solutions for our choice of parameters and functions is good. It is of interest to notice that the obtained displacements by using the second-order of the perturbation agree very well with the corresponding displacements obtained by using the method of separation of variables, a result which shows that the perturbation method is a powerful approximate method for solving the wave equation. Also, the role of the initial velocity is to adjust the resulting vibrations form being destroyed, besides it makes the time of damping of the oscillation larger. For the sake of

illustration we consider the function given by (5.82) and present the vibrations resulting from using the second-order perturbation method.

In Figures 1-30 we present the successive displacements, in ft, for $t = 0, 1, 2, \dots, 29$, respectively. It is seen from the figures that the displacement of the membrane starts to be negative after 6 seconds. After 13 seconds, the displacement starts to be positive again but with too small values. After 27 seconds the membrane approximately stopped, so that the considered damping force acts for only 27 seconds.

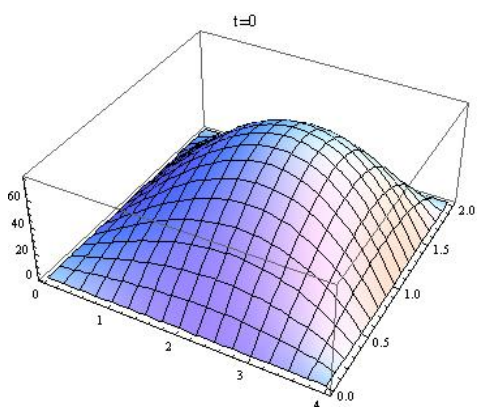


Fig.1 Displacement for $t = 0$

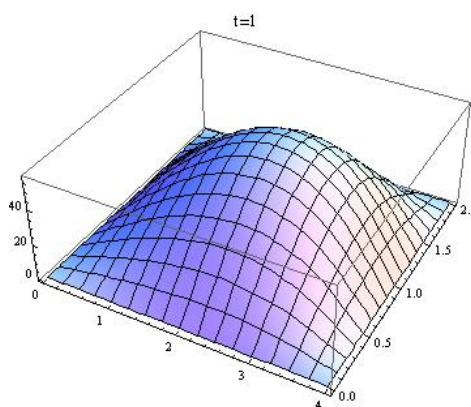


Fig.2 Displacement for $t = 1$

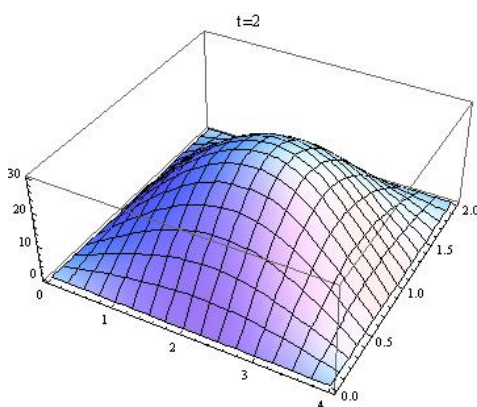


Fig.3 Displacement for $t = 2$

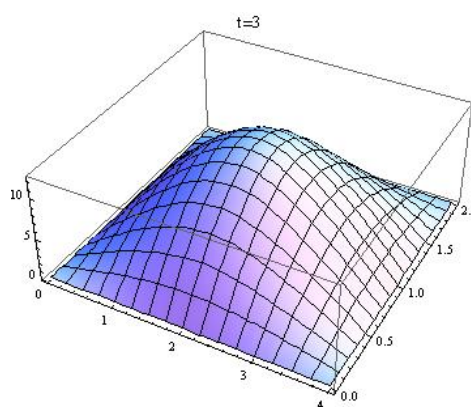
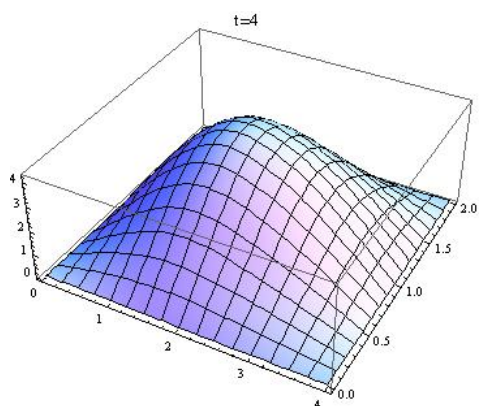
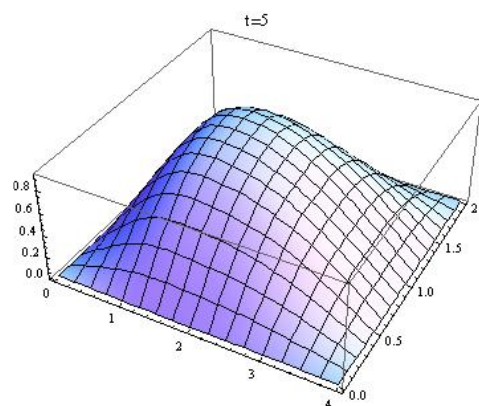
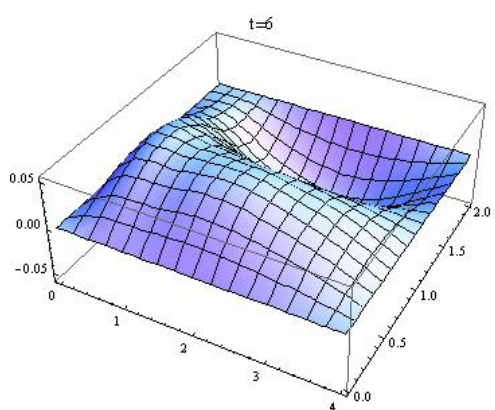
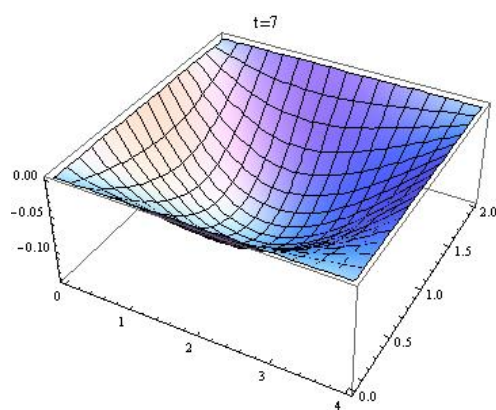
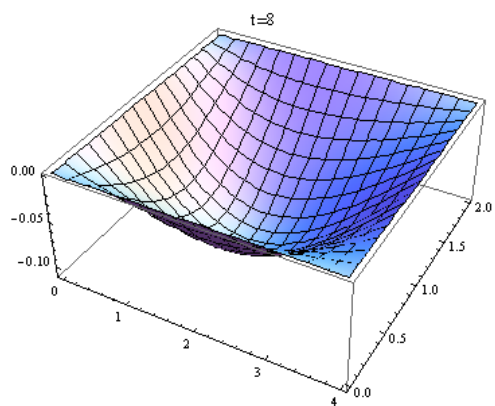
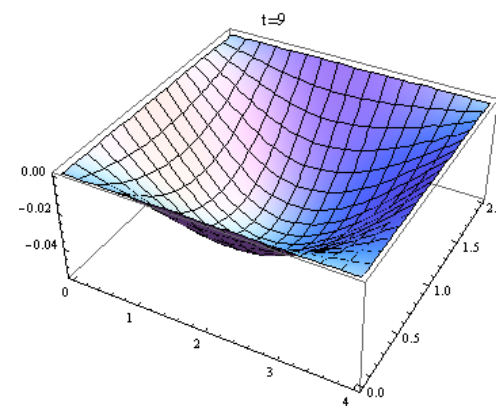
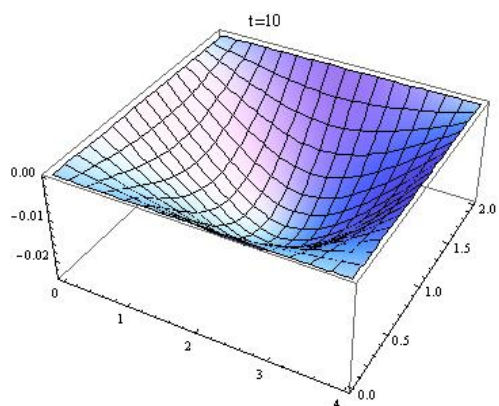
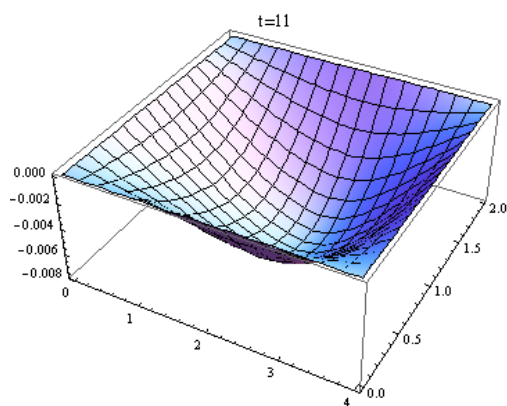
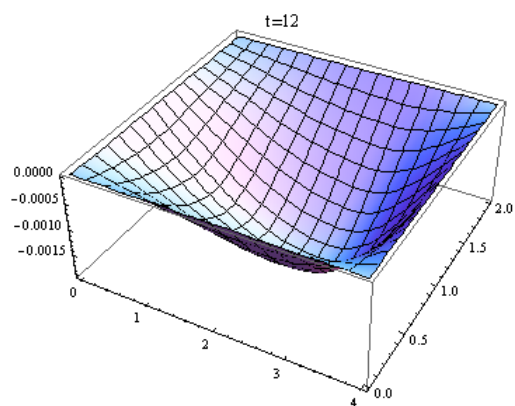
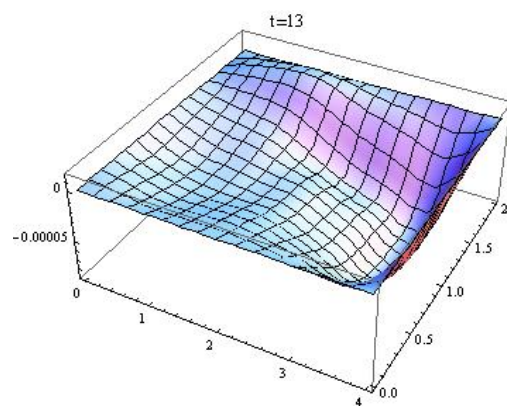
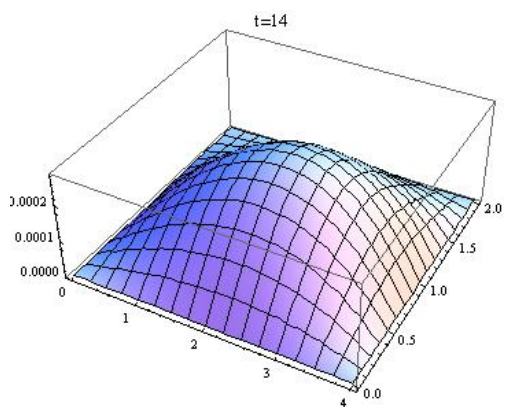
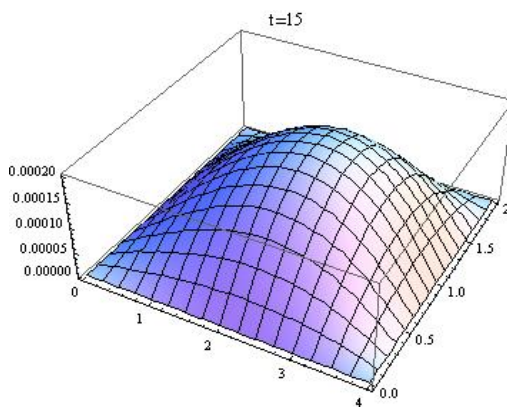
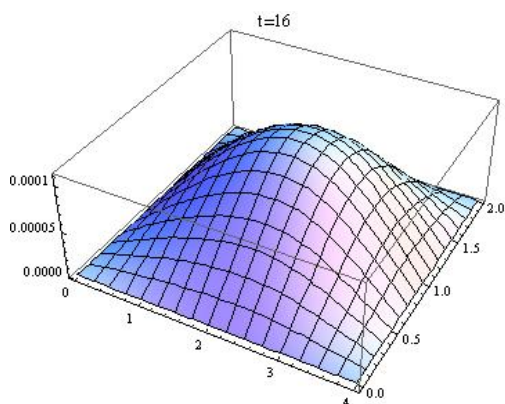
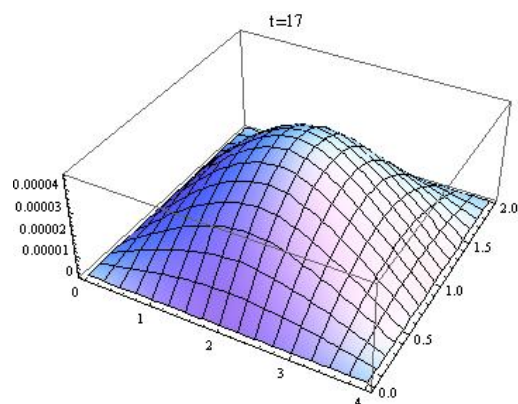
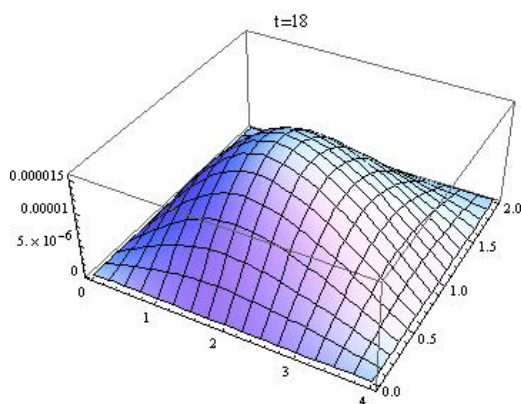
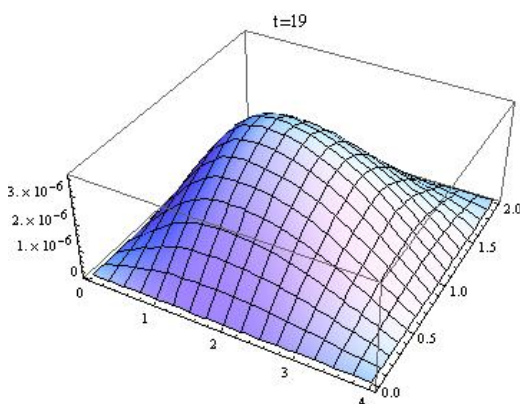
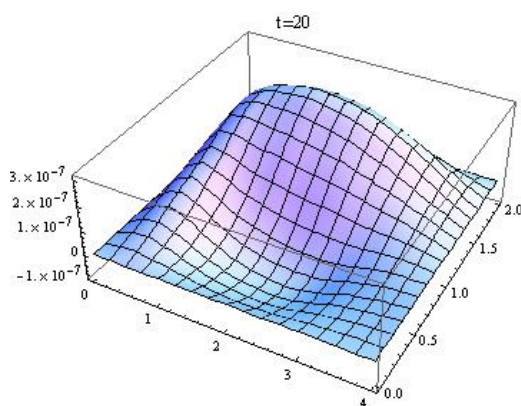
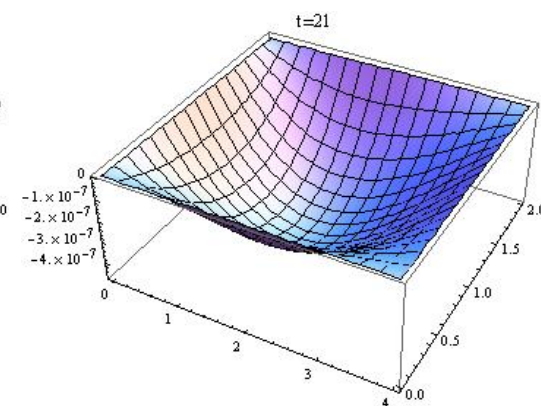
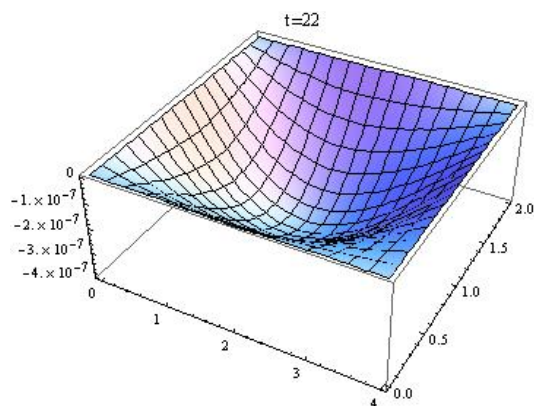
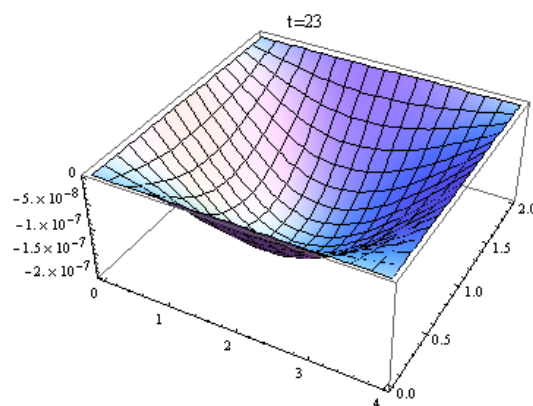
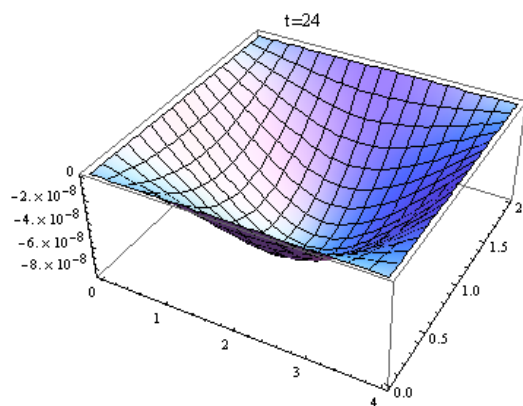
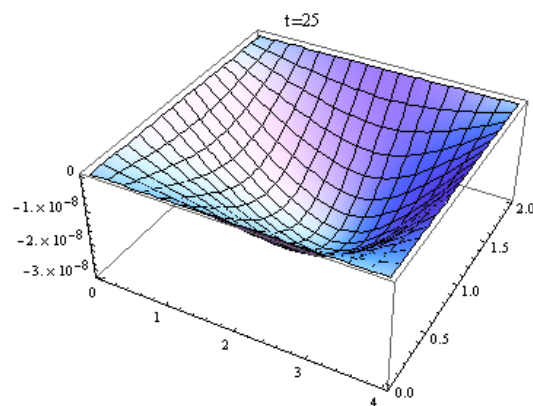
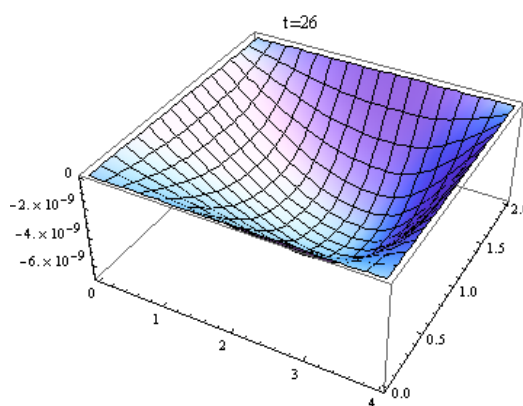
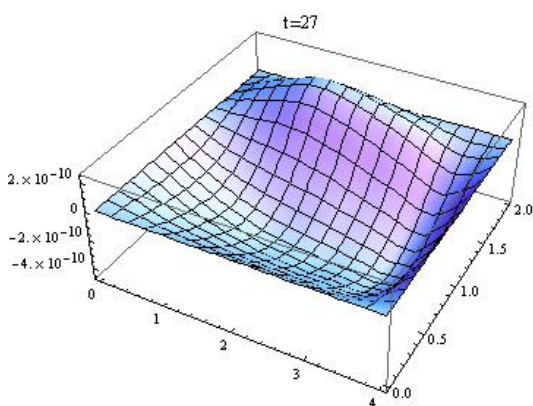


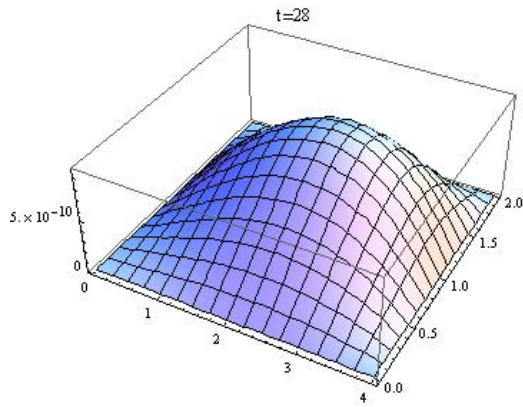
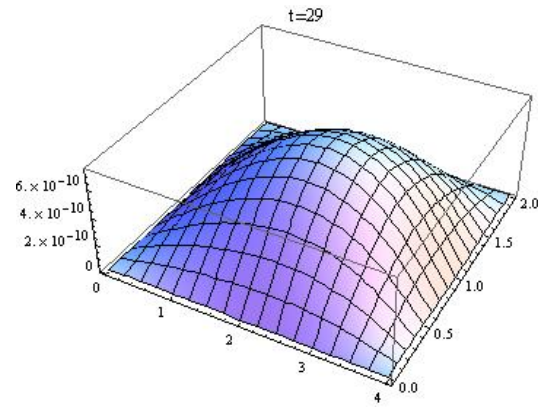
Fig.4 Displacement for $t = 3$

Fig.5 Displacement for $t = 4$ Fig.6 Displacement for $t = 5$ Fig.7 Displacement for $t = 6$ Fig.8 Displacement for $t = 7$ Fig.9 Displacement for $t = 8$ Fig.10 Displacement for $t = 9$

Fig.11 Displacement for $t = 10$ Fig.12 Displacement for $t = 11$ Fig.13 Displacement for $t = 12$ Fig.14 Displacement for $t = 13$ Fig.15 Displacement for $t = 14$ Fig.16 Displacement for $t = 15$

Fig.17 Displacement for $t = 16$ Fig.18 Displacement for $t = 17$ Fig.19 Displacement for $t = 18$ Fig.20 Displacement for $t = 19$ Fig.21 Displacement for $t = 20$ Fig.22 Displacement for $t = 21$

Fig.23 Displacement for $t = 22$ Fig.24 Displacement for $t = 23$ Fig.25 Displacement for $t = 24$ Fig.26 Displacement for $t = 25$ Fig.27 Displacement for $t = 26$ Fig.28 Displacement for $t = 27$

Fig.29 Displacement for $t = 28$ Fig.30 Displacement for $t = 29$

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