THERMAL RADIATION AND CHEMICAL REACTION EFFECTS ON UNSTEADY MHD COUETTE FLOW OF FOURTH-GRADE FLUID IN HORIZONTAL PARALLEL PLATES CHANNEL

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Abstract

This paper investigates thermal radiation and chemical reaction effects on unsteady MHD couette flow of fourth-grade fluid in horizontal parallel plates channel. The coupled governing equations are transformed into a dimensionless form using dimensionless parameters. The transformed equations are solved by implementing He-Laplace scheme explicitly. The impact of thermal radiation, chemical reaction, third and fourth-grade parameters along with diversified parameters are exhibited graphically on different flow fields. The skin friction, Nusselt number and Sherwood number are calculated, tabulated and discussed. An interesting fact is that velocity and temperature fields rise due to the increment of thermal radiation parameter. For upsurging data of chemical reaction, velocity and concentration fields diminish. Furthermore, the velocity field decline due to the increment of magnetic parameter.

Keywords: Thermal radiation, Chemical reaction, MHD, Fourth-grade fluid, He – Laplace

Nomenclature

B_0	External magnetic field.
T	Temperature of the fluid
C	Species concentration
q_r	Radiative heat flux
u	Fluid velocity
C_p	Specific heat capacity
C_f	Skin friction
N_u	Nussel number
S_h	Sherwood number
Ha	Hartmann number
P_r	Prandtl number
G_r	Grashof number due to heat transfer
G_c	Grashof number due to mass transfer
K_r	Chemical reaction parameter
S_c	Schmidt number
T_w	Temperature at the surface
T_∞	Ambient temperature as $y \to \infty$
C_w	Concentration at the surface
C_∞	Concentration as $y \to \infty$
x, y	Cartesian coordinates

Greek Symbols

μ Coefficient of shear viscosity
$\alpha_i, \beta_i, \gamma_i$ Material constants
β Thermal expansion coefficient
β_c Concentration expansion coefficient
δ Thermal radiation parameter
σ Stefan – Boltzmann constant
ρ Density of the fluid
ν Kinematic viscosity

1. Introduction

Thermal radiation is the process by which energy in the form of electromagnetic radiation, is emitted by a heated surface in all directions and travels directly to its point of absorption at the speed of light. It ranges in wavelength from the longest infrared rays through the visible – light spectrum to the ultraviolet rays. The intensity and distribution of radiant energy within this range is governed by the temperature of the emitting surface; this is in agreement to Stefan – Boltzmann law which states that "the total radiant heat emitted by a surface is proportional to the fourth power of its absolute temperature". And the rate at which a body radiates (or absorbs) thermal radiation depends upon the nature of the surface as well. Objects that are good emitter are also good absorbers (Kirchoff's radiation law).

Chemical reaction is a process that involves rearrangement of the molecular or ionic structure of a substance, as distinct from a change in physical form or nuclear reaction. There are two types of such reactions namely homogeneous reaction which occurs uniformly throughout a given phase of a flow and heterogeneous reaction which takes place in a particular region or within the boundary of a phase Umavathi [1]. The study of heat transfer with chemical reaction is of most realistic significance to engineers and scientists because of its universal incidence in many branches of science and engineering. This phenomenon plays a significant role in chemical industry, power and cooling industry for dying, evaporation, energy transfer in cooling tower and flow in s desert cooler, etc. Satya et.al [2]

Magneto hydrodynamics (MHD) is the study of the magnetic properties and behaviour of electrically conducting fluids. Example of such magneto fluids include plasma, liquid metals, salt water and electrolytes. The fundamental concept of MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. And, couette flow is the flow of a viscous fluid in the space between two surfaces, one of which is moving tangentially relative to the other. The configuration often takes the form of two parallel plates.

The fourth-grade fluid flow model is an exceptional model which has opened new subway of fluid mechanics. This sort of model is being used to explain the flow attitude of non-Newtonian fluids, which are considered vital and applicable in many industrial producing processes such in the drilling of oil and gas wells, polymer extrusion from dye, glass fibre, paper production and draining of plastics films etc.

A vast scientific analysis of non-Newtonian fluids problems has been done by many researchers. This has gained great importance in different fields due to their huge range of engineering and commercial applications. The study of the behavior of the motion of non-Newtonian fluids is very much more complicated and difficult as compared to that for Newtonian fluids, because of the nonlinear relationship between the stress and the rate of strains. The governing equations that describe the flow of Newtonian fluid is the Navier-Stokes equations, while for the flow of the non-Newtonian fluids there is no single governing equation which describes all of their properties and thus it is difficult to describe these fluids as Newtonian fluids. Therefore, many empirical and semi-empirical non-Newtonian models or constitutive equations have been proposed Islam et.al [3], where was considered the steady flow of a non-Newtonian fluid with slippage between the plate and the fluid. The constitutive equations of the fluids were modelled for fourth-grade non-Newtonian fluid with partial slip. They employed homotopy perturbation and optimal homotopy asymptotic methods to solve the non-linear differential equation.

The fourth-grade fluid has large number of complex parameters. But, Hayat et.al [4] experimented this by the effect of uniform magnetic field combined with steady and unsteady flows over porous plates. They used differential equation of order six for a numerical solution with a total three boundary conditions related with momentum and finite difference. Lie point symmetries were applied by them to reduce the number and order of distinct variable of partial differential equations.

Hayat et.al [5] examined the unsteady flow of a hydrodynamic fluid past a porous plate. The constitutive equations of the fluids were modelled by therefore a fourth-grade fluid. The study gave rise to a boundary value problem consisting of a fifth-order differential equation but there were only two boundary conditions. The solution was obtained by implementation of Lie group method. In another study, Hayat et al. [6] used Laplace transform method to determine the analytical solution of couette flows of a second grate fluid. Stokes and couette flows due to oscillating wall were discussed by Khaled and Vafai [7]. Singh [8] studied the periodic solution of oscillatory couette flow of through porous medium in rotating system. Guria [9] discussed couette flow problem for rotating and oscillatory flow.

Idowu *et al.* [10] considered the unsteady Couette flow with transpiration of a viscous fluid in a rotating system. An exact solution of the governing equations has been obtained by using Laplace Transform Technique.

Couette flow of an unsteady third grade fluid with variable magnetic field was investigated by Hayat and Kara [11], were they considered the fluid to be in an annular region between two coaxial cylinders. The axial couette flow problem of an electrically conducting fluid in an annulus was examined by Zaman et.al [12].

Unsteady MHD couette flow between two infinite parallel porous plates in an inclined magnetic field with heat transfer was studied by Joseph et al. [13]. The lower plate was considered porous. The governing equations of the flow field were solved by variable separable technique and the expression for the velocity, temperature, skin frictions and Nusselt numbers were obtained. In another investigation, Zaman et.al [14] analyzed the Couette flow problem for an unsteady magnetohydrodynamic (MHD) fourth-grade fluid in the presence of pressure gradient and Hall currents. The arising non-linear problem was solved by the homotopy analysis method (HAM).

Taha et.al [15] carried out an analysis to study the time – dependent flow of an incompressible electrically conducting fourth grade fluid over an infinite porous plate. The flow was caused by the motion of the porous plate in its own plane with an impulsive velocity $V(t)$. The governing non – linear problem was solved by invoking the Lie group theoretical approach and numerical technique. While, Taza et.al [16] studied the unsteady thin film flow of a fourth-grade fluid over a moving and oscillating vertical belt. They employed a domain decomposition method (ADM) and optimal homotopy asymptotic method (OHAM) to find the solution of the non- linear differential equations that governed the flow.

Much later, Arifuzzaman et.al [17] analyzed heat and mass transfer characteristics of naturally corrective hydro-magnetic flows of fourth grade radiative fluid resulting from vertical porous plate. They considered non-linear order chemical reaction and heat generation with thermal diffusion. The complete fundamental equations are transforming into dimensionless equations by implementing finite difference scheme explicitly.

Motivated by the above literatures, this paper presents an investigation into the thermal radiation and chemical effects on unsteady MHD couette flow of fourth-grade fluid in horizontal parallel plates channel.

2. Formulation of the Problem

We consider the unsteady flow of an electrically conducting incompressible fourth grade fluid between two horizontal parallel plates channel as shown in figure 1 below. The fluid is subjected to a uniform transverse magnetic field B_0. We assume the bottom plate is fixed (stationary) and the top plates is moving with constant velocity, u.

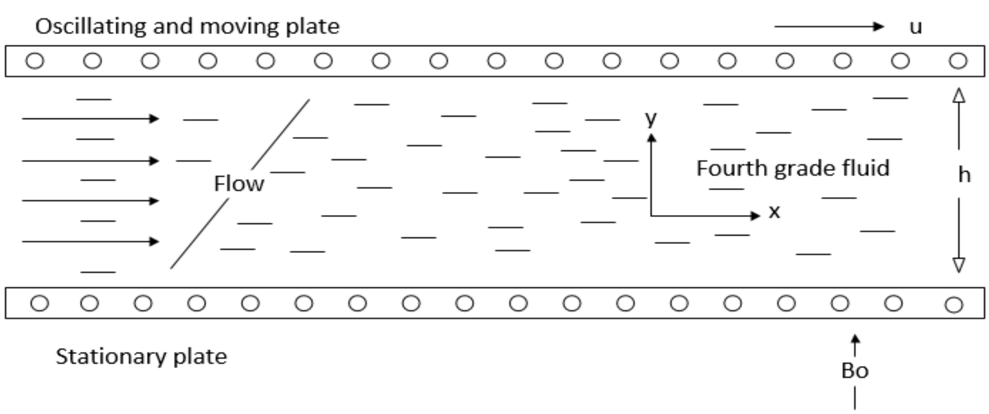

Figure 1: Physical Configuration of the Plane Couette Flow

The state of this fluid is determined by the history of the deformation gradient without a preferred reference configuration. Its constitutive equation can be written as

$$T(x,t) = -PI + f_{s=0}^{\infty}\left(F_t^t(s)\right) \tag{1}$$

where PI is the undetermined part of the stress – tensor, F is the deformation gradient and f is the functional.

Coleman and Noll [18] prescribed different sort of incompressible fluid category n as viscous fluid agreeing on Hayat et.al [4]. Incompressible fluid of differential type of grade n is the simple fluid obeying the constitutive equation

$$T = -PI + \sum_{j=1}^{n} S_j \tag{2}$$

obtained by asymptotic expansion of the functional in equation (1.1) through a retardation parameter α. For $n = 4$ as Hayat et.al ([4], [5]) and Arifuzzamn et.al [17], the first four (4) tensors S_j are given by

$$S_1 = \mu A_1 \tag{3}$$

$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2 \tag{4}$$

$$S_3 = \beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3(trA_1^2)A_1 \tag{5}$$

$$S_4 = \gamma_1 A_4 + \gamma_2(A_3 A_1 + A_1 A_3) + \gamma_3 A_2^2 + \gamma_4(A_2 A_1^2 + A_1^2 A_2) + \gamma_5(trA_2)A_2 + \gamma_6(trA_2)A_1^2 + [\gamma_7 trA_3 + \gamma_8 tr(A_2 A_1)]A_1 \tag{6}$$

Where, μ is the coefficient of shear viscosity, $\alpha_i(i = 1,2)$, $\beta_i(i = 1,2,3)$ and $\gamma_i(i = 1, \ldots, 8)$ are material constants. The A_n are the Rivlin – Ericksen tensors defined by the recursion relation

$$A_n = \frac{d}{dt}A_{n-1} + A_{n-1}L + L^T A_{n-1}, \; n>1 \tag{7}$$

$$A_1 = L + L^T \tag{8}$$

where, $L = \nabla V$, $\frac{d}{dt}$ is the material time derivative and $V$is the velocity. We note that when $\gamma_i = 0$, the fourth-grade model reduces to the third-grade model. When $\beta_i = 0$, the third-grade model reduces to second grade model. When $\alpha_i = 0$, $\beta_i = 0$ and $\gamma_i = 0$ then the model reduces to classical Navier – Stoke fluid.

The thermally radiative and chemically reactive flow is heading x – direction along infinite porous plate with heat generation. Here, U_0 is the uniform velocity, T_∞ and C_∞ are the fluid temperature and concentration.

Under the above consideration, the equations that describe the physical circumstances are;

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\alpha_1 \nu}{\rho}\frac{\partial^3 u}{\partial y^2 \partial t} + \frac{\beta_1 \nu^2}{\rho}\frac{\partial^4 u}{\partial y^2 \partial t^2} + \frac{6(\beta_2+\beta_3)}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \frac{\gamma_1 \nu^3}{\rho}\frac{\partial^5 u}{\partial y^2 \partial t^3} + \frac{2\nu(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_7+\gamma_8)}{\rho C_p}\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2 \partial t}\right] - \frac{\sigma B_0^2}{\rho C_p}u + g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty) - \frac{\nu}{k}u \tag{9}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial y} \tag{10}$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2} - K_c(C - C_\infty) \tag{11}$$

From equation (10), q_r is the radiative heat flux defined as

$$\frac{\partial q_r}{\partial y} = 4\alpha^2(T_w - T) \tag{12}$$

The initial and boundary conditions are

$$\left.\begin{array}{c} u(y,t) = e^{-yh}, T(y,t) = e^{-yh}, C(y,t) = e^{-yh} \;\; at\, t = 0\, for\;\; 0 \le y \le h \\ u(y,t) = U, T(y,t) = T_w, C(y,t) = C_w\; at\; y = h\; for\; t \ge 0 \\ u(y,t) \to \infty, T(y,t) \to \infty, C(y,t) \to \infty\; as\; y \to \infty\; for\; t > 0 \end{array}\right\} \tag{13}$$

where u is the fluid velocity, T is the temperature and C is the species concentration equation, T_w is the temperature at the surface, C_w is the concentration at the surface, T_∞ is the ambient temperature, q_r is the radiative heat flux, ρ is the density of the fluid, C_p is the heat capacity, B_0 is the external magnetic field.

In order to transform equations (9) – (13), we use the following dimensionless parameters

$$u = \frac{u^*}{U_0},\; p^* = \frac{p}{\mu h^2}, t = \frac{\nu t^*}{h^2}, G_r = \frac{g\beta(T_W^* - T_\infty^*)h^2}{\nu^2}, G_c = \frac{g\beta_T(C_W^* - C_\infty^*)h^2}{\nu^2}, Ha^2 = \frac{\sigma B_0^2 h^2}{\rho\nu}, Da = \frac{K}{h^2}, S_c = \frac{\nu}{D},\; y = \frac{y^*}{h}, x = \frac{x^*}{h}\theta = \frac{T - T_0}{T_W^* - T_\infty^*}, C = \frac{C^* - C_0^*}{C_W^* - C_\infty^*}, \delta = \frac{4\alpha^2 h^2}{\rho c_p \nu} \tag{14}$$

Substituting equation (14) into equations (9) – (13) and by dropping the asterisks, we have the following:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1\frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma_1\frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_3 u + G_r\theta + G_c C \tag{15}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{P_r}\frac{\partial^2 T}{\partial y^2} + \delta\theta \tag{16}$$

$$\frac{\partial C}{\partial t} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} - K_r C \tag{17}$$

And the initial and boundary conditions become

$$\left.\begin{array}{c} u(y,t) = e^{-y}, \theta(y,t) = e^{-y}, C(y,t) = e^{-y}\; at\; t = 0\; for\; 0 \le y \le 1 \\ u(y,t) = 1, \theta(y,t) = 1, C(y,t) = 1\; at\; y = 1\; for\; t \ge 0 \\ u(y,t) \to \infty, \theta(y,t) \to \infty, C(y,t) \to \infty\;\; as\; y \to \infty\; for\; t > 0 \end{array}\right\} \tag{18}$$

where, $l_1 = 6(\beta_2 + \beta_3), l_2 = 2(3\gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 + \gamma_8), l_3 = Ha^2 + \frac{1}{Da}$ (18a)

3. Method of Solution/Solution of the Problem

In this section we employed the He – Laplace scheme to solve equations (15) to (17) subjects to the initial and boundary conditions (18).
Since equation (15) is a coupled non – linear partial differential equation, we have to solve equations (16) and (17) first.

Now applying Laplace transform on equation (17) we have,

$$L\left\{\frac{\partial C}{\partial t}\right\} = \frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\} \tag{19}$$

Applying the initial condition and dividing through by s and rearranging we get;

$$L\{C(y,t)\} = \frac{e^{-y}}{s} - \frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\} \tag{20}$$

Taking the inverse Laplace transform of both sides of equation (20), results in

$$C(y,t) = e^{-y} + L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\}\right] \tag{21}$$

Applying the Homotopy perturbation technique, equation (21) yields

$$\sum_{n=0}^{\infty} P^n C_n(y,t) = e^{-y} + P\left[L^{-1}\left\{\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\}\right\}\right] \tag{22}$$

Comparing the coefficients of the like powers of $'P'$ in equation (22) the following approximations were obtained;

$$P^0: C_0(y,t) = e^{-y} \tag{23}$$

$$P^1: C_1(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_0}{\partial y^2}\right\} - L\{K_r C_0\}\right\}\right] = L^{-1}\left\{\frac{1}{S_c}\left(\frac{e^{-y}}{s^2}\right) - K_r\left(\frac{e^{-y}}{s^2}\right)\right\}$$

$$= \frac{e^{-y}}{S_c} t - K_r e^{-y} t \tag{24}$$

$$P^2: C_2(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_1}{\partial y^2}\right\} - L\{K_r C_1\}\right\}\right]$$

$$= L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{e^{-y}t}{S_c} - K_r e^{-y} t\right\} - L\left\{K_r\left(\frac{e^{-y}t}{S_c} - K_r e^{-y} t\right)\right\}\right\}\right]$$

$$= \frac{e^{-y}t^2}{2!S_c^2} - \frac{K_r e^{-y}t^2}{S_c} + \frac{K_r^2 e^{-y}t^2}{2!} \quad (25)$$

In viewing equations (23), (24) and (25), the solution to equation (17) is

$$C(y,t) = C_0(y,t) + C_1(y,t) + C_2(y,t) + \cdots \quad (26)$$

$$C(y,t) = e^{-y} + \frac{e^{-y}}{S_c}t - K_r e^{-y}t + \frac{e^{-y}t^2}{2!S_c^2} - \frac{K_r e^{-y}t^2}{S_c} + \frac{K_r^2 e^{-y}t^2}{2!} \quad (27)$$

Next, we consider equation (16):

$$\frac{\partial \theta}{\partial t} = \frac{1}{P_r}\frac{\partial^2 T}{\partial y^2} + \delta\theta$$

Now applying Laplace transform on equation (16) gives;

$$L\left\{\frac{\partial \theta}{\partial t}\right\} = \frac{1}{P_r}L\left\{\frac{\partial^2 \theta}{\partial y^2}\right\} - L\{\delta\theta\} \quad (28)$$

Applying the initial condition and dividing through by s and rearranging we obtain;

$$L\{\theta(y,t)\} = \frac{e^{-y}}{s} - \frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2 \theta}{\partial y^2}\right\} - L\{\delta\theta\}\right\} \quad (29)$$

Taking the inverse Laplace transform of both sides of equation (29) gives,

$$\theta(y,t) = e^{-y} + L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2 \theta}{\partial y^2}\right\} - L\{\delta C\}\right\}\right] \quad (30)$$

Applying the Homotopy perturbation technique equation (30) yields,

$$\sum_{n=0}^{\infty} P^n \theta_n(y,t) = e^{-y} + P\left[L^{-1}\left\{\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2 \theta}{\partial y^2}\right\} + L\{\delta\theta\}\right\}\right\}\right] \quad (31)$$

Comparing the coefficients of the like powers of $'P'$ in equation (31), the following approximations are obtained;

$$P^0 : \theta_0(y,t) = e^{-y} \quad (32)$$

$$P^1 : \theta_1(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2 \theta_0}{\partial y^2}\right\} + L\{\delta\theta_0\}\right\}\right] = L^{-1}\left\{\frac{1}{P_r}\left(\frac{e^{-y}}{s^2}\right) + \delta\left(\frac{e^{-y}}{s^2}\right)\right\}$$

$$= \frac{e^{-y}}{P_r}t + \delta e^{-y}t \quad (33)$$

$$P^2 : \theta_2(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2 \theta_1}{\partial y^2}\right\} + L\{\delta\theta_1\}\right\}\right]$$

$$= L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{e^{-y}t}{P_r} + \delta e^{-y}t\right\} + L\left\{\delta\left(\frac{e^{-y}t}{P_r} - K_r e^{-y}t\right)\right\}\right\}\right]$$

$$= \frac{e^{-y}t^2}{2!P_r^2} - \frac{\delta e^{-y}t^2}{P_r} + \frac{\delta e^{-y}t^2}{2!} \quad (34)$$

Therefore, in viewing equations (32), (33) and (34), the solution to equation (16) is

$$\theta(y,t) = \theta_0(y,t) + \theta_1(y,t) + \theta_2(y,t) + \cdots \quad (35)$$

$$\theta(y,t) = e^{-y} + \frac{e^{-y}}{P_r}t - \delta e^{-y}t + \frac{e^{-y}t^2}{2!P_r^2} - \frac{\delta e^{-y}t^2}{P_r} + \frac{\delta^2 e^{-y}t^2}{2!} + \cdots \quad (36)$$

Finally, we now solve equation (15):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2 \left[2 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_3 u + G_r\theta + G_c C$$

Applying the Laplace transform on both sides of equation (15) gives;

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2 \left[2 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_3 u + G_r\theta + G_c C\right\} \quad (37)$$

But,

$$L\left\{\frac{\partial u}{\partial t}\right\} = sL\{u(y,t)\} - u(y,0) \quad (38)$$

Hence,

$$L\{u(y,t)\} = \frac{u(y,0)}{s} + \frac{1}{s} L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2 \left[2 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_3 u + \frac{G_r}{s}\left(e^{-y} + \frac{e^{-y}}{P_r}t - \delta e^{-y}t + \frac{e^{-y}t^2}{2!P_r^2} - \frac{\delta e^{-y}t^2}{P_r} + \frac{\delta^2 e^{-y}t^2}{2!}\right) + \frac{G_c}{s}\left(e^{-y} + \frac{e^{-y}}{S_c}t - K_r e^{-y}t + \frac{e^{-y}t^2}{2!S_c^2} - \frac{K_r e^{-y}t^2}{S_c} + \frac{K_r^2 e^{-y}t^2}{2!}\right)\right\} \quad (39)$$

Taking the inverse Laplace transform of both sides of equation (39), we have;

$$L^{-1}\{L\{u(y,t)\}\} = L^{-1}\left\{\frac{u(y,0)}{s} + \frac{1}{s} L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2 \left[2 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_3 u + \frac{G_r}{s}\left(e^{-y} + \frac{e^{-y}}{P_r}t - \delta e^{-y}t + \frac{e^{-y}t^2}{2!P_r^2} - \frac{\delta e^{-y}t^2}{P_r} + \frac{\delta^2 e^{-y}t^2}{2!}\right) + \frac{G_c}{s}\left(e^{-y} + \frac{e^{-y}}{S_c}t - K_r e^{-y}t + \frac{e^{-y}t^2}{2!S_c^2} - \frac{K_r e^{-y}t^2}{S_c} + \frac{K_r^2 e^{-y}t^2}{2!}\right)\right\}\right\} \quad (40)$$

Or,

$$u(y,t) = e^{-y} + l_4 e^{-y}t + G_c e^{-y}t + \frac{G_r e^{-y}t^2}{2!S_c} - \frac{G_r \delta e^{-y}t^2}{2!} + \frac{G_r e^{-y}t^3}{3!P_r^2} - \frac{2G_r\delta e^{-y}t^3}{3!} + \frac{2G_c\delta^2 e^{-y}t^3}{3!} + G_c e^{-y}t + \frac{G_c e^{-y}t^2}{2!S_c} - \frac{G_c K_r e^{-y}t^2}{2!} + \frac{G_c e^{-y}t^3}{3!S_c^2} - \frac{2G_c K_r e^{-y}t^3}{3!} + \frac{2G_c K_r^2 e^{-y}t^3}{3!} + L^{-1}\left\{\frac{1}{s} L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2 \left[2 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_3 u\right\}\right\} \quad (41)$$

Applying the Homotopy perturbation method to equation (41), gives,

$$\sum_{n=0}^{\infty} P^n u_n(y,t) = e^{-y} + l_4 e^{-y}t + G_c e^{-y}t + \frac{G_r e^{-y}t^2}{2!S_c} - \frac{G_r\delta e^{-y}t^2}{2!} + \frac{G_r e^{-y}t^3}{3!P_r^2} - \frac{2G_r\delta e^{-y}t^3}{3!} + \frac{2G_c\delta^2 e^{-y}t^3}{3!} + G_c e^{-y}t + \frac{G_c e^{-y}t^2}{2!S_c} - \frac{G_c K_r e^{-y}t^2}{2!} + \frac{G_c e^{-y}t^3}{3!S_c^2} - \frac{2G_c K_r e^{-y}t^3}{3!} + \frac{2G_c K_r^2 e^{-y}t^3}{3!} + P\left(L^{-1}\left\{\frac{1}{s}\,L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} + \beta_1\frac{\partial^4 u}{\partial y^2\partial t^2} + l_1 H_a(u_n) + \gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3} + l_2\left[2H_b(u_n) + H_c(u_n)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}\right] - l_3 u\right\}\right\}\right) \quad (42)$$

Where, $H_a(u_n), H_b(u_n)$ and $H_c(u_n)$ are the He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2, \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y}$ and $\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}$ respectively.

The He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2$ are;

$$\begin{cases} H_0(u) = (u_0')^2 \\ H_1(u) = 2u_0'u_1' \\ H_2(u) = 2u_0'u_2' + (u_1')^2 \\ H_3(u) = 2u_1'u_2' \\ \vdots \end{cases} \quad (43)$$

The He's polynomials for $\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y}$ are;

$$\begin{cases} H_0(u) = u_0'''{u''}_{0t} \\ H_1(u) = u_0'''{u''}_{1t} + u_1'''{u''}_{1t} \\ H_2(u) = u_0'''{u'}_{2t} + u_1'''{u'}_{1t} + u_2'''{u'}_{0t} \\ H_3(u) = u_1'''u_{2t}'' + u_2'''u_{1t}'' \\ \vdots \end{cases} \quad (44)$$

While, the He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}$ are;

$$\begin{cases} H_0(u) = (u_0')^2(u_0''u_{0t}') \\ H_1(u) = (u_0')^2(u_0''u_{1t}') + (u_0')^2(u_1''u_{0t}') + 2u_0'u_1'(u_0''u_{0t}') \\ H_2(u) = (u_0')^2(u_0''u_{2t}') + (u_0')^2(u_1''u_{1t}') + (u_0')^2(u_2''u_{0t}') \\ \quad +2u_0'u_1'(u_0''u_{1t}') + 2u_0'u_1'(u_1''u_{0t}') + 2u_0'u_2'(u_0''u_{0t}') \\ \vdots \end{cases} \quad (45)$$

Now, comparing the like powers of "P" in equation (42) and equating their coefficients gives

$$P^0; u_0(y,t) = e^{-y} + l_4 e^{-y}t + G_c e^{-y}t + \frac{G_r e^{-y}t^2}{2!S_c} - \frac{G_r\delta e^{-y}t^2}{2!} + \frac{G_r e^{-y}t^3}{3!P_r^2} - \frac{2G_r\delta e^{-y}t^3}{3!} + \frac{2G_c\delta^2 e^{-y}t^3}{3!} + G_c e^{-y}t + \frac{G_c e^{-y}t^2}{2!S_c} - \frac{G_c K_r e^{-y}t^2}{2!} + \frac{G_c e^{-y}t^3}{3!S_c^2} - \frac{2G_c K_r e^{-y}t^3}{3!} + \frac{2G_c K_r^2 e^{-y}t^3}{3!} \quad (46)$$

$$P^1; u_1(y,t) = L^{-1}\left\{\frac{1}{s}\,L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1\frac{\partial^4 u}{\partial y^2 \partial t^2} + l_1(u_0')^2 + \gamma_1\frac{\partial^5 u}{\partial y^2 \partial t^3} + l_2[2u_0'''u''_{0t} + (u_0')^2(u_0''u_{0t}')] - l_3 u\right\}\right\} \quad (47)$$

And,

$$u_1(y,t) = \Big(e^{-y} + \alpha_1 G_c e^{-y}t + \alpha_1 G_r e^{-y}t - \beta_1 G_c K_r e^{-y} + \beta_1 G_r \delta e^{-y} - 2\gamma_1 G_c K_r e^{-y} + \gamma_1 G_c K_r^2 e^{-y} + \gamma_1 G_r \delta^2 e^{-y} + 2\gamma_1 \delta G_r e^{-y} + L_3 e^{-y} + \frac{\beta_1 G_r e^{-y}}{P_r} + \frac{\gamma_1 G_c e^{-y}}{S_c} + \frac{\gamma_1 G_r e^{-y}}{P_r^2} + L_1 e^{-2y}\Big)t + \Big(G_c e^{-y} + G_r e^{-y} + \alpha_1 G_r e^{-y} - \alpha_1 G_c K_r e^{-y} - 2\beta_1 G_c K_r e^{-y} + \beta_1 G_c K_r^2 e^{-y} + 2\beta_1 \delta G_r e^{-y} + \beta_1 \delta^2 G_r e^{-y} + \frac{\alpha_1 G_c e^{-y}}{S_c} - + \frac{\alpha_1 G_r e^{-y}}{P_r} + \frac{\beta_1 G_c e^{-y}}{S_c^2} + \frac{\beta_1 G_r e^{-y}}{P_r^2} + L_3 L_4 + L_3 G_c e^{-y} + L_3 G_r e^{-y} + 2L_1 G_c e^{-2y} + 2L_1 G_r e^{-2y}\Big)\frac{t^2}{2!} + \Big(G_r \delta e^{-y} - G_c K_r e^{-y} - 2\alpha_1 G_c K_r e^{-y} + \alpha_1 G_c K_r^2 e^{-y} + \alpha_1 \delta^2 G_r e^{-y} + \alpha_1 G_r \delta e^{-y} + \frac{G_c e^{-y}}{S_c} + \frac{G_r e^{-y}}{P_r} + \frac{\alpha_1 G_c e^{-y}}{S_c^2} + \frac{\alpha_1 G_r e^{-y}}{P_r^2} - 2L_1 G_c K_r e^{-2y} + 2L_1 G_r \delta e^{-2y} + 2L_1 G_c^2 e^{-2y} + 4L_1 G_r G_c e^{-2y} + 2L_1 G_r^2 + \frac{2L_1 G_c e^{-2y}}{S_c} + \frac{2L_1 G_r e^{-2y}}{P_r} + L_3 e^{-y} - L_3 G_c K_r e^{-y} + L_3 G_r \delta e^{-y} + \frac{L_3 G_r e^{-y}}{P_r}\Big)\frac{t^3}{3!} + \Big(2G_c K_r e^{-y} + G_r \delta^2 e^{-y} + G_r K_r e^{-y} + 2\delta G_r e^{-y} + \frac{G_c e^{-y}}{S_c} + \frac{G_r e^{-y}}{P_r^2} - 2L_1 G_c K_r e^{-2y} + 2L_1 G_c K_r^2 e^{-2y} - 2L_1 G_r \delta^2 e^{-2y} + 4L_1 G_r \delta e^{-2y} - 6L_1 G_c^2 K_r e^{-2y} + 6L_1 G_r G_c e^{-2y} + 6L_1 G_r G_c \delta e^{-2y} - 6L_1 G_r G_c K_r e^{-2y} + 6L_1 G_r^2 \delta e^{-2y} + \frac{2L_1 G_c e^{-2y}}{S_c^2} + \frac{2L_1 G_r e^{-2y}}{P_r^2} + \frac{4L_1 G_c^2 e^{-2y}}{S_c} + \frac{6L_1 G_c G_r e^{-2y}}{S_c} + L_3 G_c K_r^2 e^{-y} - L_3 G_c K_r e^{-y} + 2L_3 G_r \delta e^{-y} + L_3 G_r \delta^2 e^{-y} + \frac{L_3 G_c e^{-y}}{S_c^2} + \frac{2L_3 G_r e^{-y}}{P_r^2}\Big)\frac{t^4}{4!} + \Big(8L_1 G_c^2 e^{-2y} - 16L_1 G_c^2 K_r e^{-2y} + 14L_1 G_c^2 K_r^2 e^{-2y} - 8L_1 G_c G_r \delta e^{-2y} + 16L_1 G_c G_r \delta e^{-2y} - 12L_1 G_c G_r K_r \delta e^{-2y} - 16L_1 G_c G_r K_r e^{-2y} + 8L_1 G_c G_r K_r^2 e^{-2y} + 14L_1 G_r^2 e^{-2y} - 2L_1 G_r^2 \delta^2 e^{-2y} + 16L_1 G_r^2 \delta e^{-2y} + \frac{8L_1 G_c G_r e^{-2y}}{P_r^2} + \frac{6L_1 G_c^2 e^{-2y}}{S_c^2} - \frac{12L_1 G_c^2 K_r e^{-2y}}{S_c} + \frac{12L_1 G_c G_r e^{-2y}}{P_r S_c} + \frac{12L_1 G_c G_r \delta e^{-2y}}{S_c} - \frac{12L_1 G_c G_r K_r e^{-2y}}{P_r} + \frac{8L_1 G_c G_r e^{-2y}}{S_c^2} + \frac{12L_1 G_r^2 \delta e^{-2y}}{P_r}\Big)\frac{t^5}{5!} + \Big(20L_1 G_c^2 K_r^2 e^{-2y} + 20L_1 G_c G_r K_r \delta^2 e^{-2y} - 20L_1 G_c^2 K_r^3 e^{-2y} - 80L_1 G_c G_r K_r \delta e^{-2y} + 20L_1 G_c G_r K_r^2 \delta e^{-2y} - 20L_1 G_r^2 \delta^3 e^{-2y} + 40L_1 G_r^2 \delta^2 e^{-2y} + \frac{20L_1 G_c^2 e^{-2y}}{S_c^3} - \frac{40L_1 G_c^2 K_r e^{-2y}}{S_c} + \frac{20L_1 G_c G_r e^{-2y}}{S_c P_r^2} - \frac{20L_1 G_c G_r \delta^2 e^{-2y}}{S_c} + \frac{40L_1 G_c G_r \delta e^{-2y}}{S_c} - \frac{20L_1 G_c^2 K_r e^{-2y}}{S_c^2} - \frac{20L_1 G_c G_r K_r e^{-2y}}{P_r^2} + \frac{20L_1 G_c G_r e^{-2y}}{P_r S_c^2} + \frac{20L_1 G_c G_r \delta e^{-2y}}{S_c^2} - \frac{40L_1 G_c G_r K_r e^{-2y}}{P_r} + \frac{20L_1 G_c G_r K_r^2 e^{-2y}}{P_r} + \frac{20L_1 G_r^2 e^{-2y}}{P_r^3} - \frac{20L_1 G_r^2 \delta^2 e^{-2y}}{P_r} + \frac{40L_1 G_r^2 \delta e^{-2y}}{P_r} + \frac{20L_1 G_r^2 \delta e^{-2y}}{P_r^2}\Big)\frac{t^6}{6!} + \Big(80L_1 G_c^2 K_r^2 e^{-2y} - 80L_1 G_c^2 K_r^3 e^{-2y} + 80L_1 G_c G_r K_r \delta^2 e^{-2y} - 160L_1 G_c G_r K_r \delta e^{-2y} + 20L_1 G_c^2 K_r^4 e^{-2y} - 40L_1 G_c G_r K_r^2 \delta^2 e^{-2y} + 80L_1 G_c G_r K_r \delta e^{-2y} + 20L_1 G_r^2 \delta^4 e^{-2y} - 80L_1 G_r^2 \delta^3 e^{-2y} + 80L_1 G_r^2 \delta^2 e^{-2y} + \frac{20L_1 G_c^2 e^{-2y}}{S_c^4} - \frac{80L_1 G_c^2 K_r e^{-2y}}{S_c^2} + \frac{40L_1 G_c^2 K_r^2 e^{-2y}}{S_c^2} + \frac{40L_1 G_c G_r e^{-2y}}{P_r^2 S_c^2} - \frac{40L_1 G_c G_r \delta^2 e^{-2y}}{S_c^2} - \frac{80L_1 G_c G_r \delta e^{-2y}}{S_c^2} - \frac{80L_1 G_c G_r K_r e^{-2y}}{P_r^2} + \frac{40L_1 G_c G_r K_r^2 e^{-2y}}{P_r^2} +$$

$$\frac{20L_1G_r^2e^{-2y}}{P_r^4}-\frac{20L_1G_r^2\delta^2e^{-2y}}{P_r^2}+\frac{80L_1G_r^2\delta e^{-2y}}{P_r^2}\Big)\frac{t^7}{7!}+(2L_2G_ce^{-2y}+2L_2G_re^{-2y})t+\Big(2L_2G_r\delta e^{-2y}-2L_2G_cK_re^{-2y}+2L_2G_c^2e^{-2y}+42L_2G_cG_re^{-2y}+2L_2G_r^2e^{-2y}+\frac{2L_2G_ce^{-2y}}{S_c}+\frac{2L_2G_re^{-2y}}{P_r}\Big)\frac{t^2}{2!}+$$

$$\Big(L_2G_cK_r^2e^{-2y}-2L_2G_cK_r^2e^{-2y}+\frac{L_2G_ce^{-2y}}{S_c^2}+\frac{L_2G_re^{-2y}}{P_r^2}+L_2G_r\delta^2e^{-2y}+L_2G_r\delta e^{-2y}+\frac{2L_2G_c^2e^{-2y}}{S_c}+2L_2G_cK_re^{-2y}+\frac{3L_2G_cG_re^{-2y}}{P_r}+3L_2G_cG_re^{-2y}+L_2G_c^2e^{-2y}-L_2G_c^2K_re^{-2y}-3L_2G_cG_rK_re^{-2y}+\frac{2L_2G_cG_re^{-2y}}{S_c}+2L_2G_r^2e^{-2y}+3L_2G_cG_r\delta e^{-2y}+3L_2G_r^2\delta e^{-2y}+\frac{L_2G_r^2e^{-2y}}{P_r}\Big)\frac{t^3}{3!}+$$

$$\Big(\frac{L_2G_c^2e^{-2y}}{S_c^2}-2L_2G_c^2K_re^{-2y}-L_2G_r^2K_r^2e^{-2y}+\frac{4L_2G_rG_ce^{-2y}}{3P_r^2}+3L_2G_cG_r\delta e^{-2y}+L_2G_cG_r\delta^2e^{-2y}+\frac{L_2G_c^2e^{-2y}}{S_c}-L_2G_c^2K_re^{-2y}+\frac{L_2G_rG_ce^{-2y}}{P_r}-\frac{L_2G_c^2K_re^{-2y}}{S_c}+L_2G_cK_r^2e^{-2y}-\frac{L_2G_rG_cK_re^{-2y}}{P_r}-L_2G_rG_cK_r\delta e^{-2y}+\frac{4L_2G_c^2e^{-2y}}{3S_c^2}+\frac{L_2G_cG_re^{-2y}}{3S_c^2}-\frac{4L_2G_c^2K_re^{-2y}}{6}-\frac{4L_2G_rG_cK_re^{-2y}}{6}+\frac{2L_2G_c^2K_r^2e^{-2y}}{6}+\frac{2L_2G_rG_cK_r^2e^{-2y}}{6}-2L_2G_rG_cK_re^{-2y}+L_2G_rG_cK_r^2e^{-2y}+\frac{L_2G_r^2e^{-2y}}{P_r^2}+2L_2G_r^2\delta^2e^{-2y}+L_2G_r^2\delta e^{-2y}+\frac{L_2G_rG_ce^{-2y}}{P_rS_c}-\frac{L_2G_rG_cK_re^{-2y}}{P_r}+\frac{4L_2G_r^2e^{-2y}}{3P_r^2}+\frac{2L_2G_r^2\delta e^{-2y}}{P_r}+\frac{L_2G_rG_c\delta e^{-2y}}{S_c}-L_2G_rG_cK_r\delta e^{-2y}+\frac{L_2G_rG_c\delta^2e^{-2y}}{3}+\frac{L_2G_r^2\delta^2e^{-2y}}{3}+\frac{4L_2G_rG_c\delta e^{-2y}}{6}+\frac{4L_2G_r^2\delta e^{-2y}}{6}\Big)\frac{6t^4}{4!}+$$

$$\Big(\frac{L_2G_c^2e^{-2y}}{2S_c^2}-L_2G_c^2K_re^{-2y}+\frac{L_2G_c^2K_r^2e^{-2y}}{2}+\frac{L_2G_rG_ce^{-2y}}{2P_r^2}+\frac{L_2G_rG_c\delta^2e^{-2y}}{2}+L_2G_cG_r\delta e^{-2y}-\frac{L_2G_c^2K_re^{-2y}}{2S_c^2}+L_2G_c^2K_r^2e^{-2y}+\frac{L_2G_c^2K_r^3e^{-2y}}{2}-\frac{L_2G_rG_cK_re^{-2y}}{2P_r^2}-\frac{L_2G_rG_cK_r\delta^2e^{-2y}}{2}-L_2G_cG_rK_r\delta e^{-2y}+\frac{L_2G_c^2e^{-2y}}{3S_c^3}-\frac{L_2G_c^2K_re^{-2y}}{3S_c^2}+\frac{5L_2G_rG_ce^{-2y}}{6P_rS_c^2}+\frac{L_2G_rG_c\delta e^{-2y}}{3S_c^2}-\frac{2L_2G_c^2K_re^{-2y}}{3S_c}+\frac{2L_2G_c^2K_r^2e^{-2y}}{3}-\frac{2L_2G_cG_rK_re^{-2y}}{3}-\frac{2L_2G_cG_rK_r\delta e^{-2y}}{3}+\frac{L_2G_c^2K_re^{-2y}}{3S_c}-\frac{L_2G_c^2K_r^3e^{-2y}}{3}+\frac{L_2G_cG_rK_r^2\delta e^{-2y}}{3}-\frac{L_2G_cG_rK_re^{-2y}}{P_r}+\frac{L_2G_cG_rK_r^2e^{-2y}}{2P_r}+\frac{L_2G_r^2e^{-2y}}{2P_r^3}+\frac{5L_2G_r^2\delta^2e^{-2y}}{6P_r}+\frac{4L_2G_r^2\delta e^{-2y}}{3P_r}+\frac{2L_2G_cG_r\delta e^{-2y}}{2S_c^2}-L_2G_cG_rK_r\delta e^{-2y}+\frac{L_2G_cG_rK_r^2\delta e^{-2y}}{2}+\frac{L_2G_r^2\delta e^{-2y}}{2P_r^2}-\frac{L_2G_r^2\delta^3e^{-2y}}{2}-L_2G_r^2\delta^2e^{-2y}+\frac{L_2G_rG_ce^{-2y}}{3P_r^2S_c}-\frac{L_2G_cG_rK_re^{-2y}}{3P_r^2}+\frac{L_2G_r^2e^{-2y}}{3P_r^3}+\frac{L_2G_rG_c\delta^2e^{-2y}}{3S_c}-\frac{L_2G_cG_rK_r\delta^2e^{-2y}}{3}+\frac{L_2G_r^2\delta^3e^{-2y}}{3}+\frac{2L_2G_cG_r\delta e^{-2y}}{3S_c}-\frac{2L_2G_cG_rK_r\delta e^{-2y}}{3}+\frac{2L_2G_r^2\delta^2e^{-2y}}{3P_r}+\frac{2L_2G_r^2\delta^2e^{-2y}}{3P_r}\Big)\frac{24t^5}{5!}+$$

$$\Big(\frac{L_2G_c^2e^{-2y}}{6S_c^4}-\frac{L_2G_c^2K_re^{-2y}}{3S_c^2}+\frac{L_2G_c^2K_r^2e^{-2y}}{6S_c^2}+\frac{L_2G_rG_ce^{-2y}}{3P_r^2S_c^2}+\frac{L_2G_rG_c\delta^2e^{-2y}}{3S_c^2}+\frac{L_2G_cG_r\delta e^{-2y}}{3S_c^2}-\frac{L_2G_c^2K_re^{-2y}}{3S_c^2}-\frac{L_2G_c^2K_r^2e^{-2y}}{3S_c^2}-\frac{2L_2G_cG_rK_re^{-2y}}{3P_r^2}-\frac{2L_2G_cG_rK_r\delta^2e^{-2y}}{3}-\frac{2L_2G_cG_r\delta e^{-2y}}{3}+\frac{L_2G_c^2K_r^2e^{-2y}}{6S_c^2}-\frac{L_2G_c^2K_r^3e^{-2y}}{3}+\frac{L_2G_c^2K_r^4e^{-2y}}{6}+\frac{L_2G_cG_rK_r^2\delta^2e^{-2y}}{3}+\frac{L_2G_cG_rK_r^2e^{-2y}}{3P_r^2}+\frac{2L_2G_cG_rK_r^2\delta e^{-2y}}{3}+\frac{L_2G_r^2e^{-2y}}{6P_r^4}+\frac{L_2G_r^2\delta^2e^{-2y}}{3P_r^2}+\frac{2L_2G_r^2\delta e^{-2y}}{3P_r^2}+\frac{L_2G_r^2\delta^4e^{-2y}}{6}+\frac{2L_2G_r^2\delta^3e^{-2y}}{3}+\frac{2L_2G_cG_rK_r\delta e^{-2y}}{3}+\frac{2L_2G_cG_rK_r\delta e^{-2y}}{3}+\frac{2L_2G_r^2\delta^2e^{-2y}}{3}\Big)\frac{120t^6}{6!}+\cdots \quad (48)$$

The skin friction C_f, the Nusselt number N_u and Sherwood numbers S_h are given as

$$C_f=\frac{\partial u}{\partial y}\Big|_{y=0},\ N_u=\frac{\partial\theta}{\partial y}\Big|_{y=0},\ S_h=\frac{\partial C}{\partial y}\Big|_{y=0} \quad (49)$$

4. Results and Discussion

Theoretical work on Couette flow of fourth-grade fluid has been investigated. The study has been analyzed on thermally radiative and chemically reactive convective fourth-grade fluid in a horizontal parallel plates channel. The impact of thermal radiation, chemical reaction, third – grade and fourth – grade parameters along with diversified physical parameters are depicted graphically on different flow fields using MATLAB. The default values for the pertinent parameters are taken as $\alpha = 0.3, \beta_1 = 1, \beta_2 = 1, \beta_3 = 1, \gamma_1 = 0.3, \gamma_2 = 0.3, \gamma_4 = 0.3, \gamma_5 = 0.3, \gamma_7 = 0.3, \gamma_8 = 0.3, S_c = 1, G_r = 5, G_c = 5, P_r = 0.71, Ha = 1, Da = 1, K_r = 1, t = 0.5$. In addition, for advanced visualization of fluid, streamlines and isotherms are also exhibited. The interaction of electrically conducting fluids with magnetic fields, through electromagnetic forces called Lorentz forces. Strong magnetic parameter Ha creates retarding force namely Lorentz force which diminishes fluid velocity.

To validate the present work; when $G_r = 0$ and $G_c = 0$, then our results would be in agreement with Zaman et al. [14]. The impression of system parameters on skin friction C_f, Nusselt number N_u and Sherwood number S_h are also investigated and presented in tables.

Figures 2 and 3 depict the velocity and temperature fields for increment of thermal radiation parameter δ $(1 \leq \delta \geq 4)$. Thermal radiation is known as electromagnetic radiation or the conversion of thermal energy which generates the thermal motion of particles in matter. Thermal radiation could be attributed due to thermal excitation. Both velocity and temperature fields are affected significantly with increase in thermal radiation parameter (δ). Thermal radiation for a medium which contains it inevitably has pressure and density gradients, and the treatment requires the use of hydrodynamics.

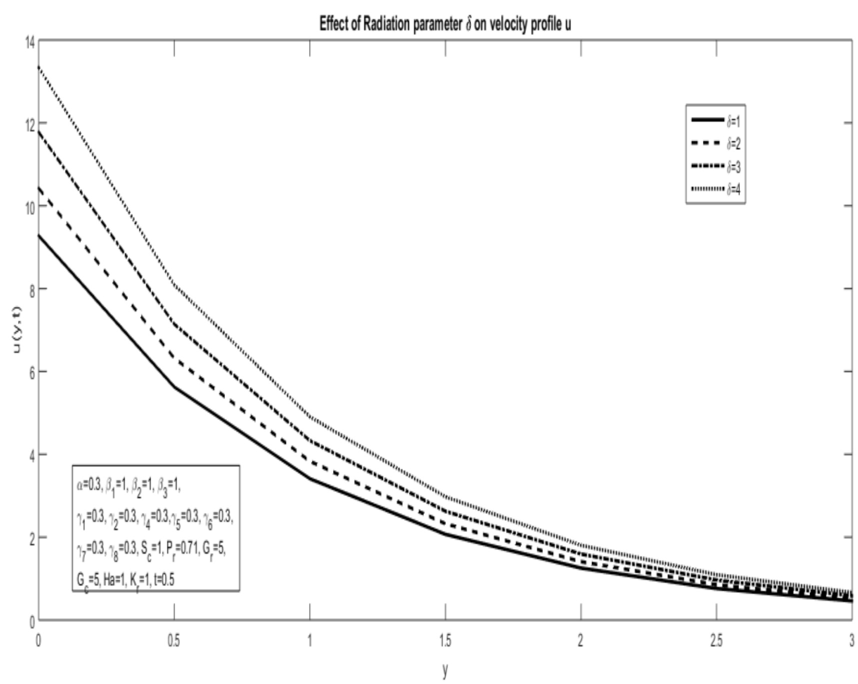

Figure 2: Effect of δ on $u(y, t)$

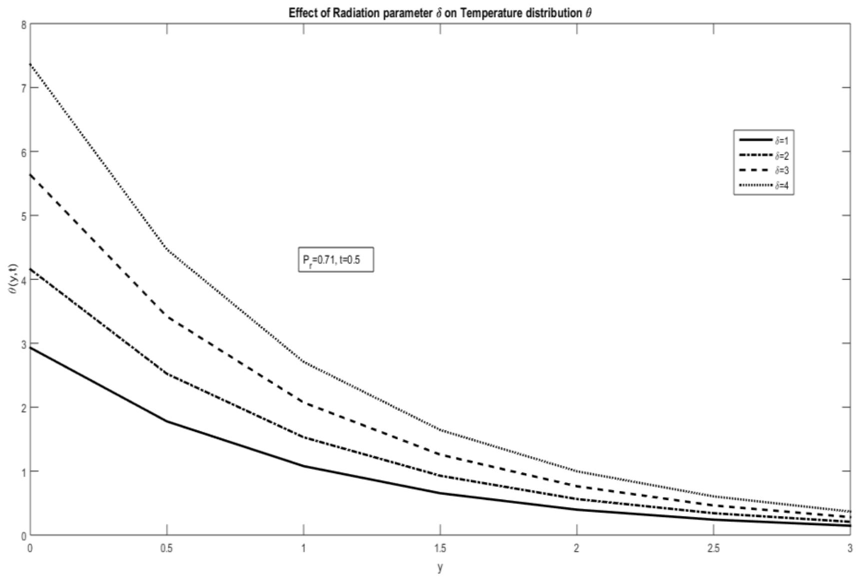

Figure 3: Effect of δ on $\theta(y,t)$

The effect of chemical reaction parameter (K_r) on velocity and concentration profiles are depicted in figures 4 and 5 respectively. Due to the rise of chemical reaction (K_r) from $1 \leq K_r \geq 4$, the velocity field decreases, and the concentration field also decreases. Physically, chemical reaction occurs with more disturbance which develops the molecular motion and upsurges the heat transport phenomena, as a result retards the velocity of the flow.

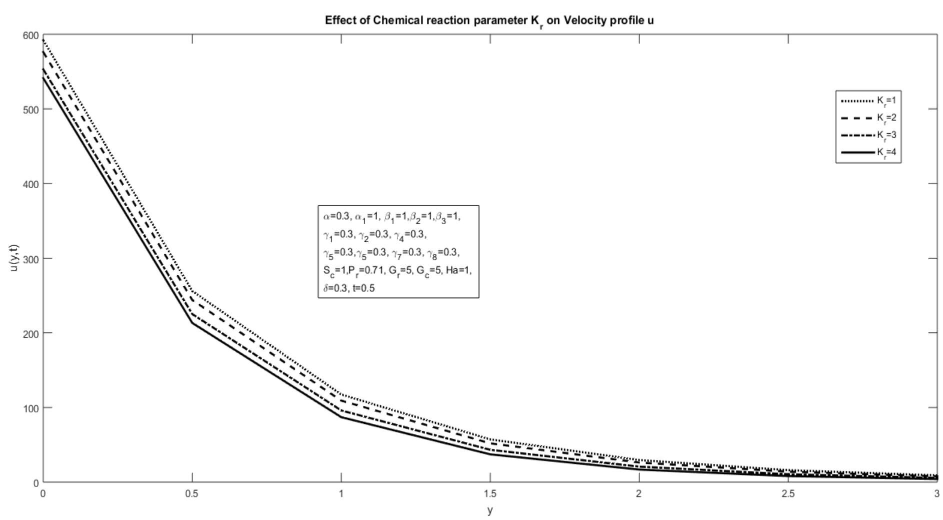

Figure 4: Effect of K_r on $u(y,t)$

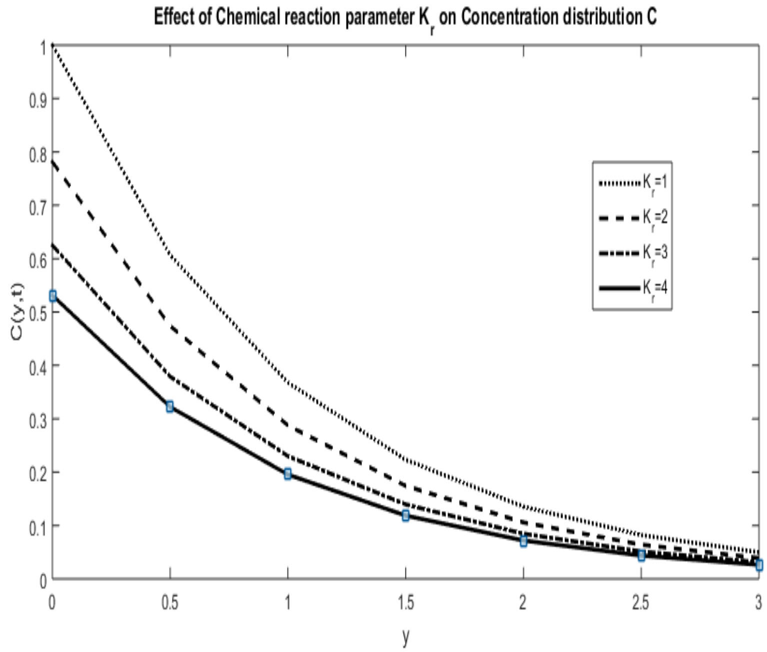

Figure 5: Effect of K_r on $C(y,t)$

Figures 6 and 7 depict the effect of Grashof number due to heat transfer (G_r) and Grashof number due to mass transfer (G_c) on velocity field respectively for the increment ($1 \leq G_r \geq 4; 1 \leq G_c \geq 4$). It is observed that the velocity field increases significantly. To this effect, at higher Grashof numbers, the flow at the boundary is turbulent, while at lower Grashof numbers, the flow at the boundary is laminar.

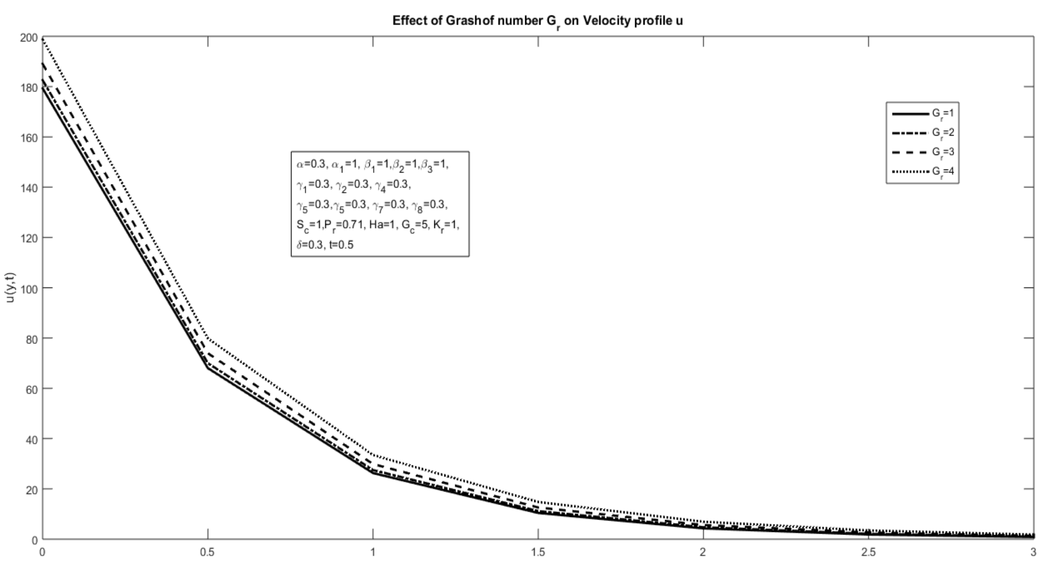

Figure 6: Effect of G_r on $u(y,t)$

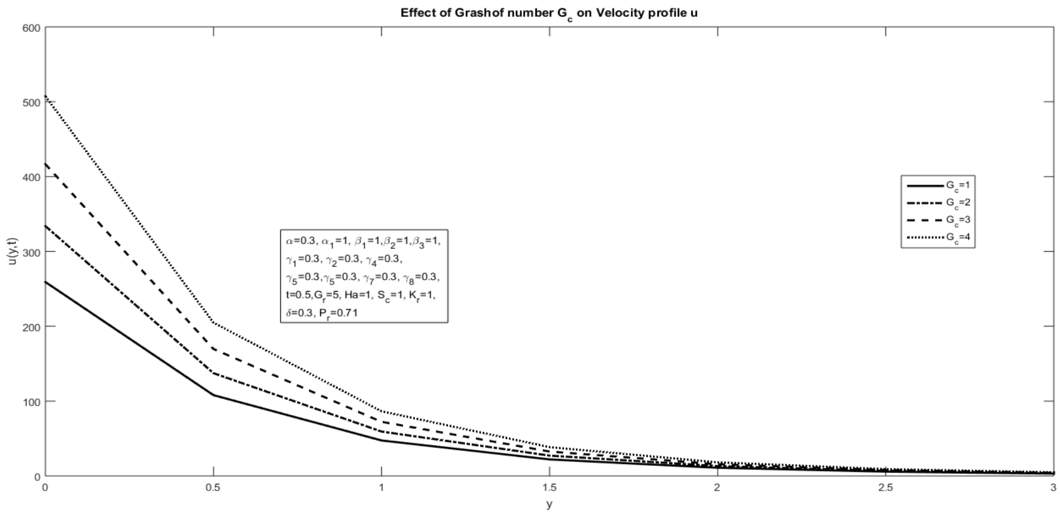

Figure 7: Effect of G_c on $u(y,t)$

Figure 8 illustrates the drag force effect on fluid flow. The velocity profile decreases with the increment of Hartmann number $(1 \leq Ha \geq 4)$. The role of Hartmann number which is the magnetic parameter is to suppress turbulence. Physically, when magnetic field is applied to any fluid, the apparent viscosity of the fluid increases to the point of becoming viscous elastic solid. It is of great interest that yield stress of the fluid can be controlled very accurately through variation of the magnetic field intensity. The result is that the ability of the fluid to transmit force can be controlled with help of electromagnet which give rise to many possible control – based applications, including MHD power generation, electromagnetic casting of metals, MHD propulsion etc.

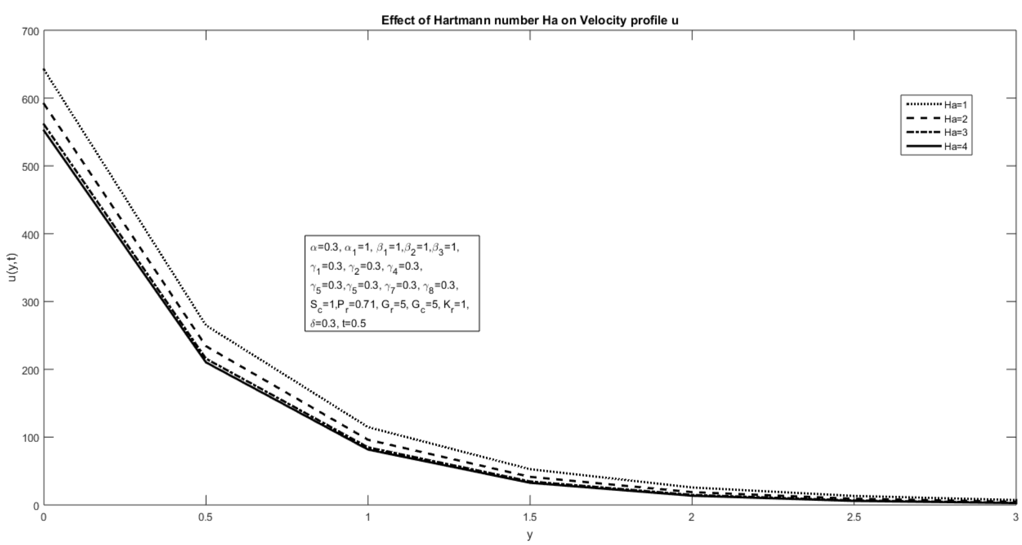

Figure 8: Effect of Ha on $u(y,t)$

Figures 9 and 10 display the velocity and concentration profiles respectively for the increment of Schmidt number $(1 \leq S_c \geq 4)$. Both the velocity and concentration profiles decrease with increase of Schmidt number (S_c). Physically, Schmidt number (S_c) helps to develop fluid concentration and concentration buoyancy force. Furthermore, it can also be used to improve the visualization of fluid fields.

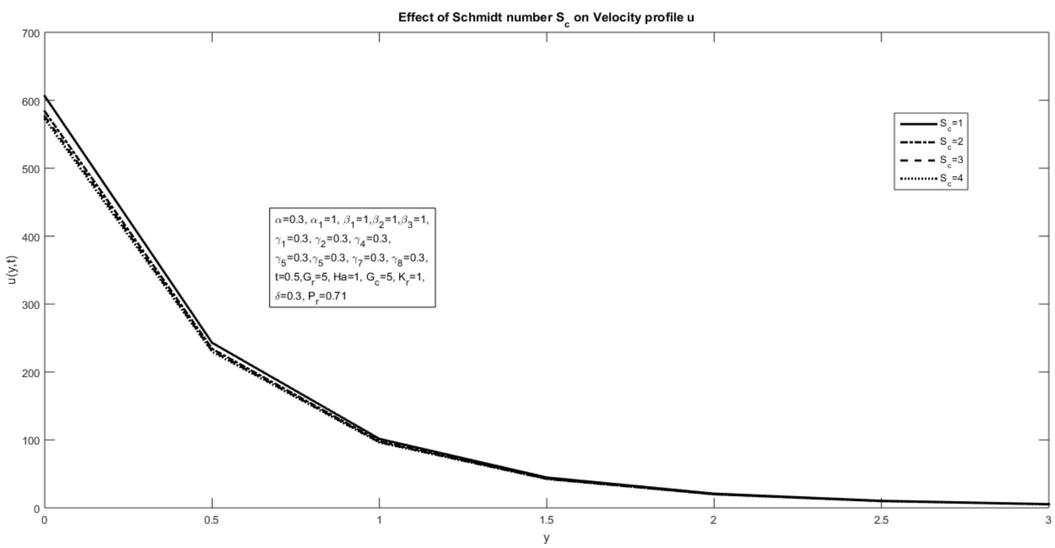

Figure 9: Effect of S_c on $u(y,t)$

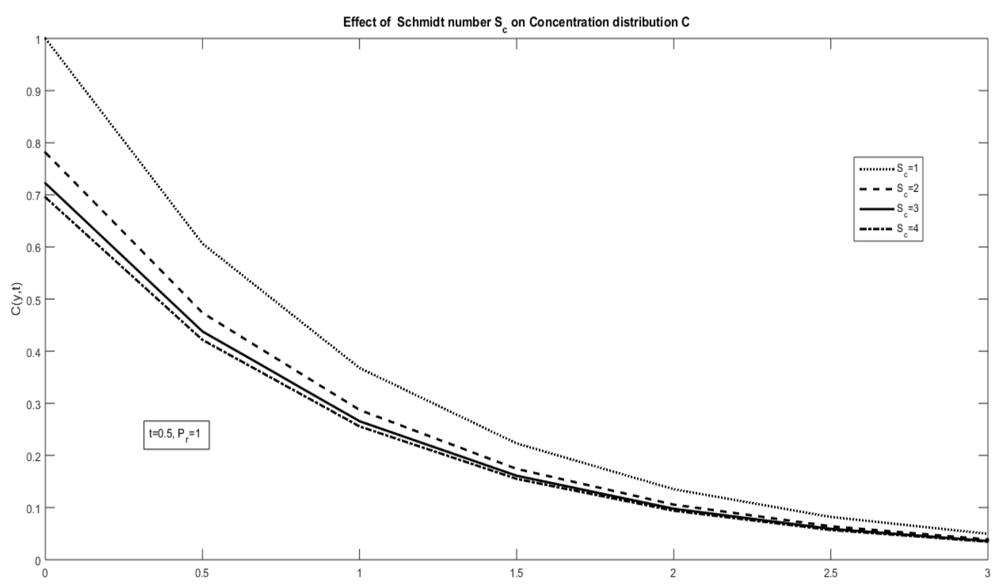

Figure 10: Effect of S_c on $C(y,t)$

Figures 11 and 12 show the impression of Prandtl number (P_r) on velocity and temperatures profiles respectively. The parameter (P_r) is the proportion of kinematic viscosity and thermal diffusivity which changes physically with temperature. For example, water $P_r = 7.0$ (at 20^0C) and Ammonia $P_r = 1.38$ decline more rapidly than air $P_r = 0.71$. However, increase in P_r depict the domination of thermal and momentum diffusivity respectively. Prandtl number is used to determine whether heat transport occurs with either conduction or convection process. Since, Prandtl number is inversely proportional to thermal diffusivity so that increasing P_r led to the decrease in velocity and temperature profiles.

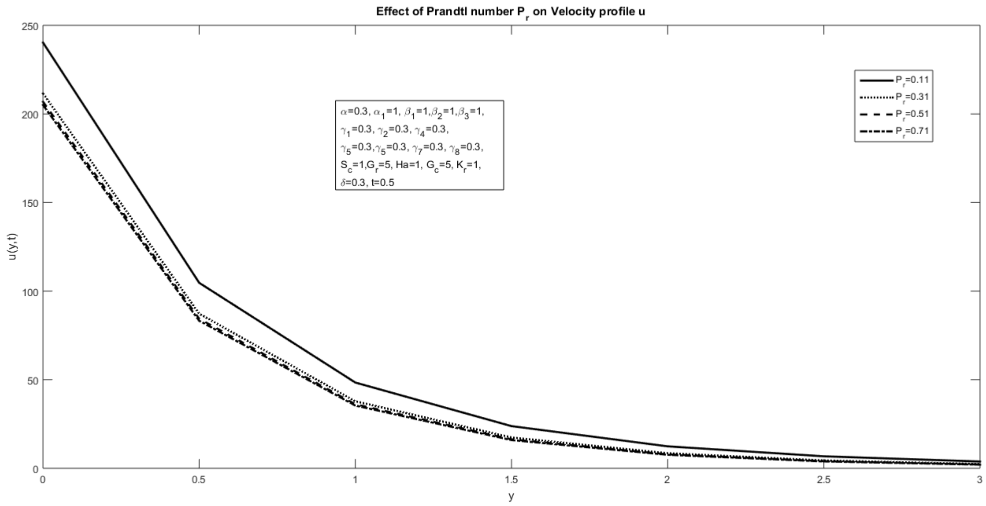

Figure 11: Effect of P_r on $u(y,t)$

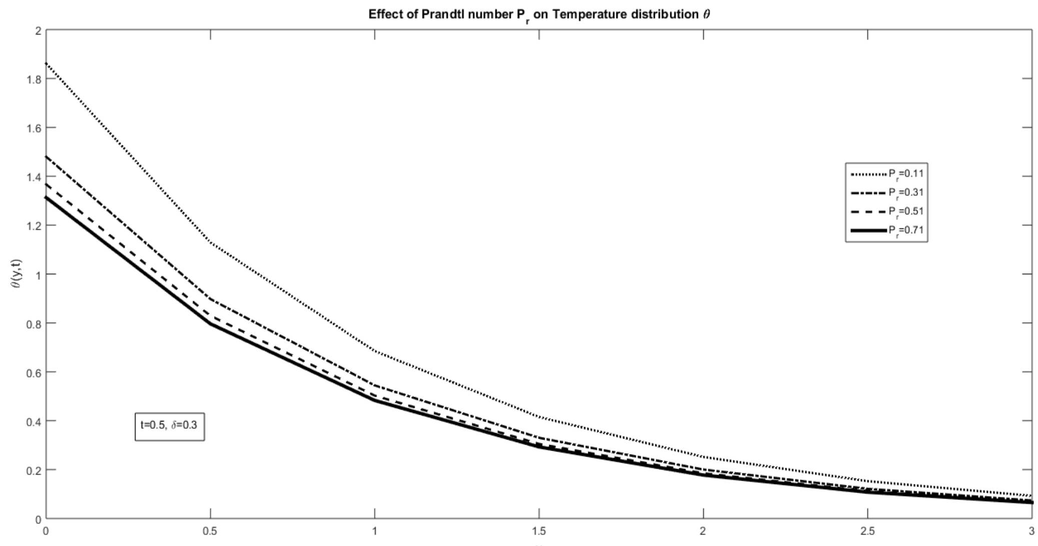

Figure 12: Effect of P_r on $\theta(y,t)$

The impression of third – grade and fourth – grade parameters on velocity profiles are respectively illustrated in figure figures 13 and 14. It is observed that the velocity profile increases with increase in both third – grade and fourth – grade parameters.

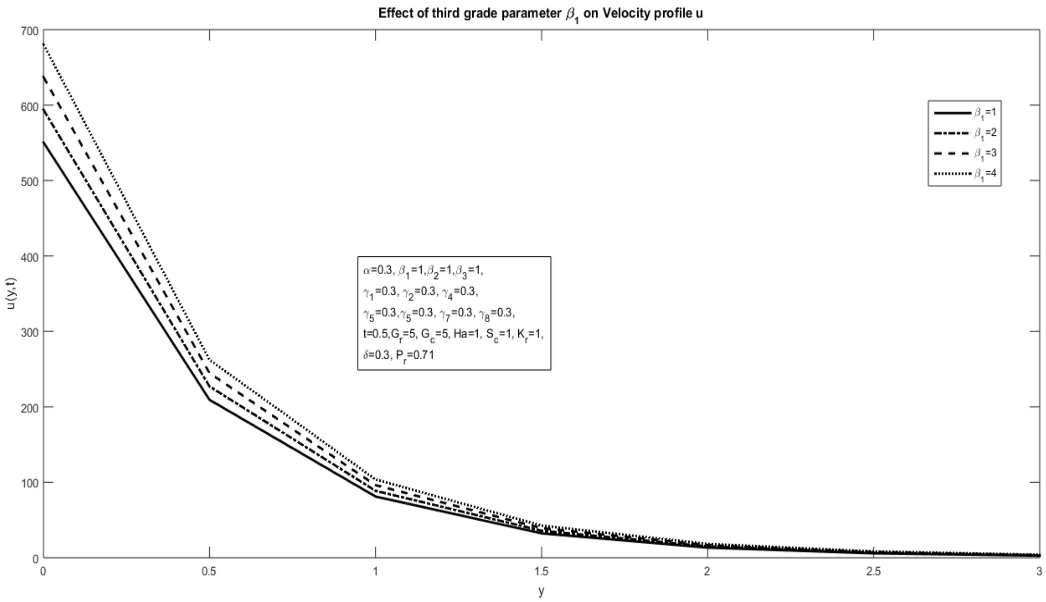

Figure 13: Effect of β_1 on $u(y,t)$

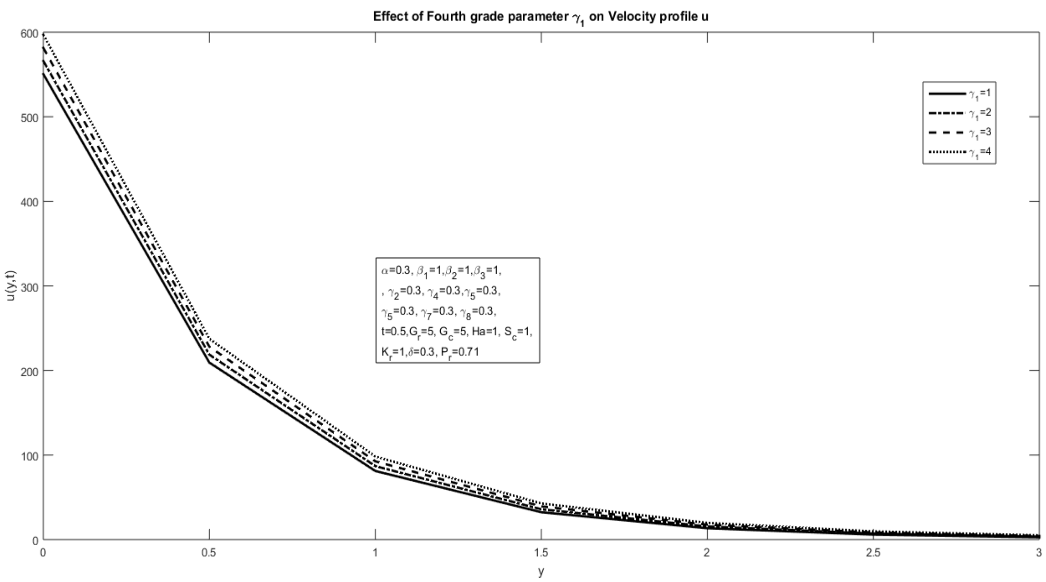

Figure 14: Effect of γ_1 on $u(y,t)$

Figures 15, 16 and 17 depict the effect of time t on velocity profile. This is to show the unsteady state of the flow fields. It is observed that both the velocity and temperature fields increases with increment of time $(0.25 \leq t \geq 1.0)$. While, the concentration field decreases with increase in time.

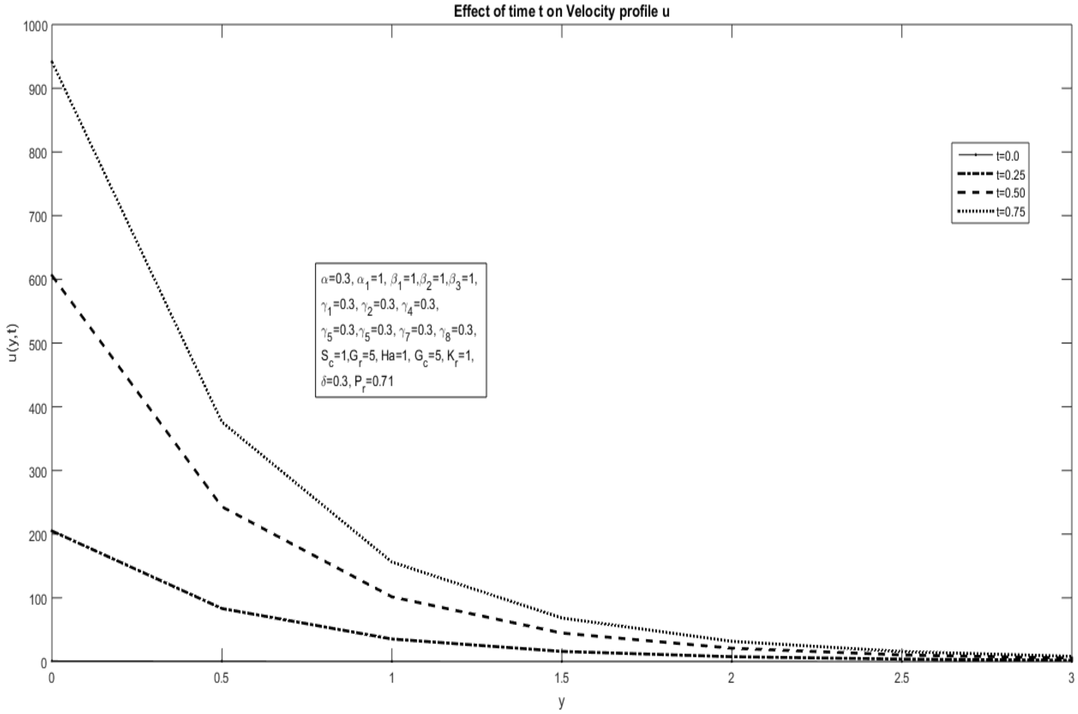

Figure 15: Effect of t on $u(y,t)$

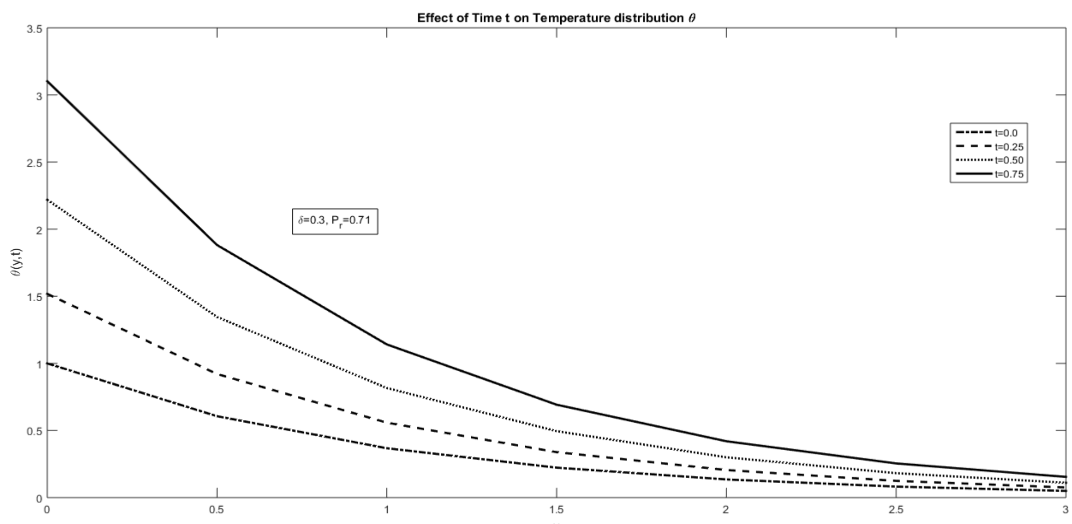

Figure 16: Effect of t on $\theta(y,t)$

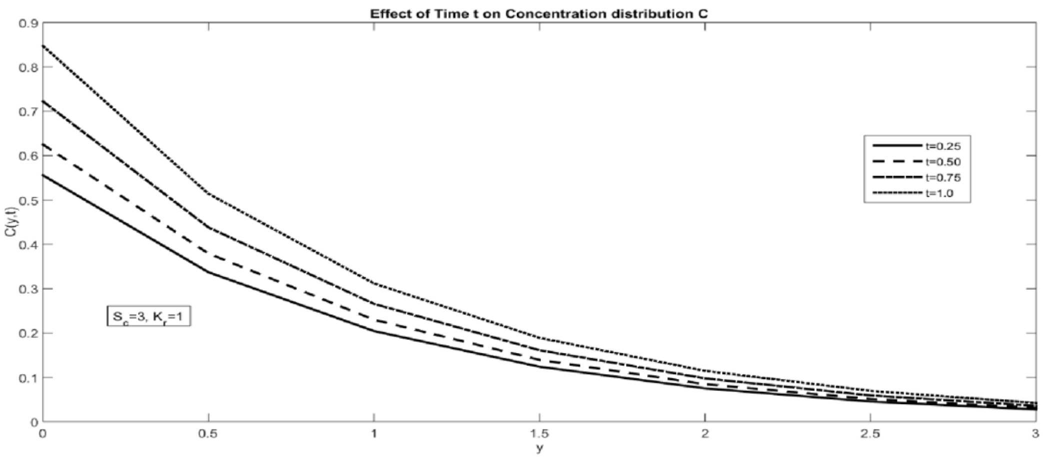

Figure 17: Effect of t on $C(y,t)$

Table 1 shows the computational values of skin friction (C_f) for flow parameters. It can be seen that C_f plunge with rising data of Ha because the magnetic field tends to decelerate the fluid flows and hence the surface declines. Increase in δ, K_r, G_c and G_r increase the field of C_f. While, increase in S_c and P_r decrease the skin friction (C_f). Table 2 illustrates the computational values of Nusselt number (N_u) with the variation of δ and P_r. It is deduced that the Nusselt number increases with increase in δ and decreases with increase in P_r. The computational values of Sherwood number S_h with varaiation of K_r and S_c are shown in table 3. It is seen that S_h increase with increase in K_r and decreases with increase in S_c.

Table 1: Computational values of Skin friction (C_f) for variation of flow parameters

δ	C_f	K_r	C_f	Ha	C_f	S_c	C_f	P_r	C_f	G_c	C_f	G_r	C_f
1	1.4465	1	1.3952	1	1.5136	1	1.5136	0.11	2.3415	1	1.0537	1	1.0569
2	1.5279	2	1.4575	2	1.5134	2	1.4812	0.31	1.6584	2	1.1687	2	1.1710
3	1.6189	3	1.5293	3	1.5132	3	1.4710	0.51	1.5542	3	1.2836	3	1.2852
4	1.7195	4	1.6107	4	1.5130	4	1.4659	0.71	1.5136	4	1.3986	4	1.3994

Table 2: Computational values of Nusselt number (N_u) for variation of flow parameters

δ	N_u	P_r	N_u
1	0.6561	0.11	2.2454
2	0.8149	0.31	0.8351
3	0.9967	0.51	0.6346
4	1.2016	0.71	0.5586

Table 3: Computational values of Sherwood number (S_h) for variation of flow parameters

K_r	S_h	S_c	S_h
1	0.5518	1	0.5518
2	0.6553	2	0.5087
3	0.7817	3	0.4956
4	0.9312	4	0.4893

5. Conclusion

Thermal radiation and chemical reaction effects on unsteady magnetohydrodynamic (MHD) couette flow of fourth-grade fluid in a horizontal parallel plates channel has been investigated. The solution for fourth-grade fluid in a horizontal parallel plates channel eith thermal radiation, chemical reaction along with diversified physical parameters has been analysed. The key findings are given below;

- Velocity and temperature fields rise due to the increment of thermal radiation parameter.
- For upsurging data of chemical reaction, velocity and concentration fields diminish.
- Velocity profile goes up when third and fourth-grade parameters get to raise.
- Velocity and skin friction fields decline due to the increment of magnetic parameter.
- Increasing Prandtl number tend to diminish the velocity and temperature profiles.
- Strong values of Schmidt number decrease the boundary layer of the Sherwood number field.
- Increase in Grashof numbers accelerate the velocity field.
- Nusselt number distribution rise due to the enhancement in thermal radiation
- Strong values of thermal radiation parameter, chemical reaction parameter and Grashof numbers increase the skin friction while higher values of Schmidt and Prandtl numbers diminish the skin friction.
- As time increases, the velocity and temperature fields increase, while the concentric field decreases.

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