Fractional Modified Bessel Functions in View of M-Truncated Local Fractional Derivative

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Abstract

The main objective of this paper is to investigate the fractional modified Bessel functions (FMBFs) via a recently proposed local fractional derivative, known as M-truncated derivative. The generating function is obtained. The power series expansion is defined and used to derive and prove some important recurrence relations. The M-truncated fractional modified Bessel differential equation is proposed, and one of its fractional power series solutions about its regular singular point $x = 0$ is found. Finally, the orthogonality relation of such functions is established and proved analytically in view of the M-truncated integral. To the best of our knowledge, this paper is the first to study FMBFs and derive their standard properties within the context of local fractional calculus.

1. Introduction

Fractional calculus owes its origin to the question raised by L'Hospital in 1695 of whether the derivative to an integer order n could be extended and still be valid when n is not an integer. At the beginning, this field of research has been only presented purely, and until very recently, researchers have realized the powerful applicability of this field in modeling many physical phenomena much better than using the ordinary usual calculus due to several properties in fractional calculus that can provide a good explanation of physical behavior of certain phenomena [1]. For example, fractional derivatives provide a better description of the model of a nonlinear oscillation of an earthquake [2], and modelling fluid dynamics with fractional derivatives can eliminate the deficiency caused by the occurrence of a continuous flow of traffic [3,4]. Therefore, fractional calculus becomes a more convenient tool for the description of mathematical models in different aspects of physical and dynamical systems [5-8]. Fractional order derivatives in the sense of Riemann-Liouville and Caputo [9] were the main tools to achieve the fourth-mentioned results, and defined as follows

Definition 1.1 The Riemann-Liouville derivative of fractional order α of function $f(t)$ is given as

$$ {}_{RL}\mathcal{D}_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (1) $$

where $n - 1 \le \alpha < n \in z^+$.

Definition 1.2 The Caputo derivative of fractional order α of function $f(t)$ is defined as

$$ {}_{C}\mathcal{D}_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(m)}(\tau) d\tau, \tag{2} $$

where $n - 1 \leq \alpha < n \in z^{+}$.

However, since the Riemann–Liouville and Caputo fractional derivative formulas are integral in form, they do not satisfy some classical features such as the chain rule, the product, and quotient of two functions. To overcome this problem, in 2014 Khalil et al. [10] proposed a local definition of fractional derivative known as a "conformable derivative," depending on the basic limit definition of derivative and defined as follows:

Definition 1.3 For the initial real value a, the conformable fractional derivative $\mathcal{D}_{a}^{\alpha} f(x)$ of a real function $f : [a, \infty) \to R$, $\alpha \in (0,1]$ is defined by the following relation:

$$ \mathcal{D}_{a}^{\alpha} f(x) = \lim_{h \to 0} \frac{f\left(x + h(x-a)^{1-\alpha}\right) - f(x)}{h}, \text{ for all } x > a. \tag{3} $$

The usability of the conformable derivative notion has wide areas of interest in both theoretical and practical aspects [11,12]. Maxwell's equations have been considered in the conformable fractional setting to describe electromagnetic fields of media in [13]. The conformable differential equation has been used for the description of the sub-diffusion process in [14]. Also, some applications in quantum mechanics have been treated in the context of conformable fractional derivatives [15]. Furthermore, in 2014, Katugampola [16] has also proposed an alternative fractional derivative with classical properties, which refers to the Leibniz and Newton calculus and is like the conformable fractional derivative. In 2018, Sousa and Oliveira [17] introduced a truncated M-fractional derivative type that unifies the existing local fractional derivatives mentioned above and which also satisfies the classical properties of integer-order calculus.

Special functions [18] play a critical role in mathematical physics, fractional calculus, the theory of differential equations, quantum mechanics, approximation theory, and many other branches of science. They have a long history that can be traced back to the past three centuries, when the problems of terrestrial and celestial mechanics, the boundary value problems of electromagnetism, and the eigenvalue problems of quantum mechanics were solved. At some level in quantum mechanics and electromagnetism, spherical harmonics arise in problems with spherical symmetry [19], where they play the role of cosines and sines in the Fourier expanding of functions. The German astronomer F. W. Bessel (1784–1846) is credited with deriving the differential equation bearing his name and carrying out the first systematic study of the general properties of its solutions (now called Bessel functions) in his famous 1824 memoir. Nonetheless, Bessel functions were first discovered in 1732 by D. Bernoulli (1700–1782), who provided a series solution (representing a Bessel function) for the oscillatory displacements of a heavy hanging chain. Bessel functions arise naturally when modeling problems with spherical and cylindrical symmetry. Because of their close association with cylindrical domains, they are called cylinder functions and belong to the family of cylinder functions, as are Hankel functions, Kelvin's functions, and Lommel functions, among others. Bessel functions can be obtained by separating the wave equation [20]

$$ \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{4} $$

in spherical coordinates, they arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder, among numerous other areas of theoretical physics. There are a host of related functions also belonging to the general family of cylinder functions, the most notable of which are the modified Bessel functions, which are most clearly distinguished by their non-oscillatory behavior. For this reason, they often appear in applications that are different in nature from those for the standard Bessel functions. For more on Bessel and modified Bessel functions, we refer the reader to the excellent book by Luke [21].

A resurgence of interest has occurred in the study of Bessel functions in the framework of fractional calculus theory [22, 23]. The study of a Bessel function of half-integer order led to the discovery of another interesting class of orthogonal polynomials called the Bessel polynomials. Many authors have used these polynomials. For example, Yüzbaşi et al. [24] solved linear integral, differential, and integro-differential equations, while Parand et al. [25] applied Bessel functions to solve nonlinear Lane-Emden equations. In [26], fractional optimal control problems were solved using the Bessel collocation method.

2. Preliminaries

In the following, we are going to mention some essential definitions and results about the M-truncated derivative.

Definition 2.1 suppose $f:[l,\infty)\to R$ and $x>0$. Also $0<\alpha<1$ and $\beta>0$, the M-truncated derivative [17] of g of order α is denoted by ${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}$, and defined by

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)=\lim_{\epsilon\to 0}\frac{f\left(t\ {}_{i}E_{\beta}(\epsilon x^{-\alpha})\right)-f(x)}{\epsilon}, \tag{5}$$

$\forall\ x>0$ and ${}_{i}E_{\beta}(.)$, $\beta>0$ is a truncated Mittag-Leffler function of one parameter defined as

$${}_{i}E_{\beta}(z)=\sum_{k=0}^{i}\frac{z^{k}}{\Gamma(\beta k+1)}. \tag{6}$$

Note that, if f is α-differentiable in some open interval $(0,a)$, $a>0$, and $\lim_{x\to 0^{+}}\left({}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)\right)$ exist, then we have

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(0)=\lim_{t\to 0^{+}}\left({}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)\right). \tag{7}$$

Theorem 2.1 [17] suppose $0<\alpha\le 1$, $\beta>0$, $l,m\ \in R$ and g, h are α-differentiable at $x>0$. Then, the M-truncated derivative meets the following properties

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}(lg+mh)(x)=l\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}g(x)+m\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}h(x); \tag{8}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}(g*h)=g(x)\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}h(x)+h(x)\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}g(x); \tag{9}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(\frac{g}{h}\right)(x)=\frac{h(x)\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}g(x)-g(x)\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}h(x)}{h(x)^{2}}; \tag{10}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}(k) = 0, \text{ where } k \text{ is a constant.} \tag{11}$$

If f is differentiable, then

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\big(f(x)\big) = \frac{x^{1-\alpha}}{\Gamma[\beta+1]}\frac{df(x)}{dx}. \tag{12}$$

The M-truncated derivative for f in Riemann-Liouville form [27] is defined as follows

Definition 2.2 suppose f is continuous and M-differentiable on (l,m) with order α, β. Then, we have

$${}_{0}^{M-RL}\mathfrak{D}_{t}^{\alpha,\beta,\gamma}\big(f(t)\big) = \frac{1}{\Gamma(n-\gamma)}\, {}_{i}^{M}\mathfrak{D}^{\alpha,\beta} \int_0^t \frac{f(\tau)}{(t-\tau)^{\gamma-n+1}}d\tau, \tag{13}$$

$$n-1<\alpha,\gamma\le n, \beta>0,$$

where ${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\big(f(t)\big) = \frac{t^{1-\alpha}}{\Gamma[\beta+1]}\frac{df(t)}{dt}$, and ${}_{i}E_{\beta}(.)$, $\beta>0$ is truncated Mittag-Leffler function of one parameter.

Definition 2.3 [17] Let $a\ge 0$ and $x\ge a$. Also, let f be a function defined in $(a,x]$ and $0<\alpha<1$. Then, the M-fractional integral of order α of a function f is defined by

$${}_{M}T_{a}{}^{\alpha,\beta}f(x) = \Gamma(\beta+1)\int_a^x \frac{f(t)}{t^{1-\alpha}}dt, \tag{14}$$

With $\beta>0$.

Theorem 2.2 [17] Let $a\ge 0$ and $0<\alpha<1$. Also, let f be a continuous function such that exist ${}_{M}T_{\alpha}{}^{\alpha,\beta}f$. Then

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left({}_{M}T_{a}{}^{\alpha,\beta}f(x)\right) = f(x), \tag{15}$$

with $x\ge a$ and $\beta>0$.

3. Fractional modified Bessel functions

3.1 Generating function and recurrence relations

Definition 3.1.1 The power series expansion of the fractional Bessel and modified Bessel functions of the first kind are defined, respectively, by

$$J_p^{\alpha}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+p+1)}\left(\frac{x^{\alpha}}{2\alpha}\right)^{p+2k}, \tag{16}$$

$$I_p^{\alpha}(x) = \sum_{k=0}^{\infty}\frac{i^{2(\alpha+1)k}}{k!\Gamma(k+p+1)}\left(\frac{x^{\alpha}}{2\alpha}\right)^{p+2k}, \tag{17}$$

with p is a real number, and $0 < \alpha < 1$.

Lemma 3.1.1 The fractional Bessel and modified Bessel functions of the first kind are related by

$$I_p^\alpha(x) = i^{-\alpha p} J_p^\alpha(ix), \tag{18}$$

with p is a real number, and $0 < \alpha < 1$.

Proof. Consider the right-hand sight of Eq. (3),

$$i^{-\alpha p} J_p^\alpha(ix) = i^{-\alpha p} \sum_{k=0}^{\infty} \frac{(-1)^k i^{(p+2k)\alpha}}{k!\Gamma(k+p+1)} \left(\frac{x^\alpha}{2\alpha}\right)^{p+2k} = \sum_{k=0}^{\infty} \frac{i^{2(\alpha+1)k}}{k!\Gamma(k+p+1)} \left(\frac{x^\alpha}{2\alpha}\right)^{p+2k} = I_p^\alpha(x). \blacksquare$$

Theorem 3.1.1 Let n be an integer. Then the generating function of the fractional modified Bessel functions of the first kind is

$$G^\alpha(x,t) = \sum_{n=-\infty}^{\infty} i^{2n(\alpha+1)} I_n^\alpha(x) t^n = e^{\frac{i^{2n(\alpha+1)} x^\alpha}{2\alpha}\left(t - \frac{1}{i^{2\alpha} t}\right)}. \tag{19}$$

Proof. First, we will find the generating function of the fractional Bessel functions (FBFs), then we will derive the generating function of the FMBFs from it by means of Lemma 3.1.1 as follows:

$$e^{\frac{x^\alpha}{2\alpha}\left(t-\frac{1}{t}\right)} = e^{\frac{x^\alpha t}{2\alpha}} e^{\frac{x^\alpha}{2\alpha t}} = \sum_{j=0}^{\infty} \frac{x^{\alpha j} t^j}{(2\alpha)^j j!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha k}}{(2\alpha)^k t^k k!} = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k! j!} \left(\frac{x^\alpha}{2\alpha}\right)^{j+k} \right) t^{j-k}.$$

Our goal is to obtain a single series in powers of t. Thus, we make the change of index $n = j - k$, consequently

$$e^{\frac{x^\alpha}{2\alpha}\left(t-\frac{1}{t}\right)} = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x^\alpha}{2\alpha}\right)^{n+2k} \right) t^n = \sum_{n=-\infty}^{\infty} J_p^\alpha(x) t^n.$$

Replacing x by ix and t by $-i^\alpha t$ in the equation above and using Eq. (18), we get

$$G^\alpha(x,t) = e^{\frac{i^{2n(\alpha+1)} x^\alpha}{2\alpha}\left(t - \frac{1}{i^{2\alpha} t}\right)} = \sum_{n=-\infty}^{\infty} i^{2n(\alpha+1)} I_p^\alpha(x) t^n. \blacksquare$$

Theorem 3.1.2 The FMBFs $I_p^\alpha(x)$ satisfy the following recurrence relations:

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(x^{\alpha p} I_p^\alpha(x)\right) = \frac{1}{\Gamma(\beta+1)} x^{\alpha p} I_{p-1}^\alpha(x); \tag{20}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(x^{-\alpha p} I_p^\alpha(x)\right) = \frac{i^{2(\alpha+1)}}{\Gamma(\beta+1)} x^{-\alpha p} I_{p+1}^\alpha(x); \tag{21}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(I_p^\alpha(x)\right) = \frac{1}{\Gamma(\beta+1)}\left[I_{p-1}^\alpha(x) - \frac{\alpha p}{x^\alpha} I_p^\alpha(x)\right] \tag{22}$$

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(I_p^{\alpha}(x)\right) = \frac{1}{\Gamma(\beta+1)}\left[\frac{\alpha p}{x^{\alpha}}I_p^{\alpha}(x) + i^{2\alpha+1}I_{p+1}^{\alpha}(x)\right]; \tag{23} $$

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(I_p^{\alpha}(x)\right) = \frac{1}{2\Gamma(\beta+1)}\left[I_{p-1}^{\alpha}(x) + i^{2(\alpha+1)}I_{p+1}^{\alpha}(x)\right]; \tag{24} $$

$$ \frac{2\alpha p}{x^{\alpha}}I_p^{\alpha}(x) = I_{p-1}^{\alpha}(x) - i^{2(\alpha+1)}I_{p+1}^{\alpha}(x). \tag{25} $$

Proof

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\eta}\left(x^{\alpha p}J_p^{\alpha}(x)\right) = {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2\alpha)^{p+2k}k!\Gamma(k+p+1)}(x^{\alpha})^{2(p+k)} = $$
$$ \frac{x^{\alpha p}}{\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+p)}\left(\frac{x^{\alpha}}{2\alpha}\right)^{p+2k-1} = \frac{x^{\alpha p}}{\Gamma(\beta+1)}J_{p-1}^{\alpha}(x). $$

Replacing x by ix in the equation above and using Eq. (18), we have

$$ \frac{1}{i^{\alpha}}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(i^{\alpha p}x^{\alpha p}J_p^{\alpha}(ix)\right) = \frac{i^{\alpha p}}{\Gamma(\beta+1)}x^{\alpha p}J_{p-1}^{\alpha}(ix), $$

From which we get

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(x^{\alpha p}I_p^{\alpha}(x)\right) = \frac{1}{\Gamma(\beta+1)}x^{\alpha p}I_{p-1}^{\alpha}(x). $$

Now to obtain Eq. (21), we proceed as follows:

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(x^{-\alpha p}J_p^{\alpha}(x)\right) = {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2\alpha)^{p+2k}k!\Gamma(k+p+1)}(x^{\alpha})^{2k} = $$
$$ \frac{1}{\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2\alpha)^{p+2k-1}(k-1)!\Gamma(k+p+1)}(x^{\alpha})^{2k-1}. $$

If we make the change of index $k = \Omega + 1$, we will have

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(x^{-\alpha p}J_p^{\alpha}(x)\right) = \frac{-x^{-\alpha p}}{\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{(-1)^{\Omega}}{\Omega!\Gamma(p+\Omega+2)}\left(\frac{x^{\alpha}}{2\alpha}\right)^{k+2\Omega+1} = \frac{i^2x^{-\alpha p}}{\Gamma(\beta+1)}J_{p+1}^{\alpha}(x), $$

replacing x by ix and applying Eq. (18), we get

$$ \frac{1}{i^{\alpha}}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(i^{-\alpha p}x^{-\alpha p}J_p^{\alpha}(ix)\right) = \frac{i^{2-\alpha p}x^{-\alpha p}}{\Gamma(\beta+1)}J_{p+1}^{\alpha}(ix), $$

from which we obtain

$$ {}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\left(x^{-\alpha p}I_p^{\alpha}(x)\right) = \frac{i^{2(\alpha+1)}}{\Gamma(\beta+1)}x^{-\alpha p}I_{p+1}^{\alpha}(x). $$

Eq. (22) is obtained from Eq. (20) by applying the M-truncated derivative. Similarly, Eq. (23) is derived from Eq. (21) by applying the M-truncated derivative. Eq. (24) is obtained by adding Eqs. (22,23), and Eq. (25) is obtained by subtracting Eqs. (22,23). ■

3.2 M-truncated fractional modified Bessel differential equation

Analogous to second order homogeneous linear differential equation

$$P(x)f''(x)+Q(x)f'(x)+R(x)f(x)=0, \tag{26}$$

we formulate the second order homogeneous linear fractional differential equation as

$$P(x)\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}f(x)+Q(x)\mathcal{D}^{\alpha}f(x)+R(x)f(x)=0. \tag{27}$$

Definition 3.2.1 A point $x=x_o$ is called α-regular singular point of Eq. (27) if:

$$\lim_{x\to x_o}(x-x_o)^{\alpha}Q(x) \text{ exists and } \lim_{x\to x_o}(x-x_o)^{2\alpha}R(x) \text{ exists.}$$

Remark 3.2.1 If P, Q and R are polynomials with no common factors, then the singular points of Eq. (27) are those for which $P(x)=0$.

Consider the M-truncated fractional modified Bessel differential equation

$$x^{2\alpha}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)+\frac{\alpha}{\Gamma(\beta+1)}x^{\alpha}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)+\frac{1}{[\Gamma(\beta+1)]^2}(i^{2\alpha}x^{2\alpha}-\alpha^2p^2)f(x)=0, \tag{28}$$

where $\alpha\in(0,1]$, p is a real number, and

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)=\frac{x^{1-\alpha}}{\Gamma(\beta+1)}f'(x), \tag{29}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)=\frac{1}{[\Gamma(\beta+1)]^2}\left(x^{2(1-\alpha)}f''(x)+(1-\alpha)x^{1-2\alpha}f'(x)\right). \tag{30}$$

If we apply the limit $i\to 0$ on both sides of Eq. (5) and take $\beta=\alpha=1$, then Eq. (28) is the classical modified Bessel differential equation.

$x=0$ is a α-regular singular point for Eq. (28). In this case, for $x>0$, to find its first solution, we write the fractional Frobenius series as follows

$$f(x)=\sum_{k=0}^{\infty}a_kx^{(k+\delta)\alpha}, \tag{31}$$

Where δ is any real number. The first and second fractional derivatives for Eq. (31) are

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)=\frac{\alpha}{\Gamma(\beta+1)}\sum_{k=0}^{\infty}(k+\delta)a_kx^{(k+\delta-1)\alpha}, \tag{32}$$

$${}_{i}^{M}\mathfrak{D}^{\alpha,\beta}\,{}_{i}^{M}\mathfrak{D}^{\alpha,\beta}f(x)=\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\sum_{k=0}^{\infty}(k+\delta)(k+\delta-1)a_kx^{(k+\delta-2)\alpha}. \tag{33}$$

Substituting Eqs. (31,32,33) into Eq. (28), we get

$$\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\sum_{k=0}^{\infty}(k+\delta)(k+\delta-1)a_kx^{(k+\delta-2)\alpha}+\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\sum_{k=0}^{\infty}(k+\delta)a_kx^{(k+\delta)\alpha}+\frac{1}{[\Gamma(\beta+1)]^2}(i^{2\alpha}x^{2\alpha}-\alpha^2p^2)\sum_{k=0}^{\infty}a_kx^{(k+\delta)\alpha}=0 \tag{34}$$

which can be reformulated as

$$\left(\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\delta(\delta-1)+\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\delta-\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2p^2\right)a_ox^{\delta\alpha}+\left(\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\delta(\delta+1)+\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2(\delta+1)-\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2p^2\right)a_1x^{(\delta+1)\alpha}+\sum_{k=2}^{\infty}\left(\left(\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2(k+\delta)(k+\delta-1)+\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2(k+\delta)-\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2p^2\right)a_k+\frac{i^{2\alpha}a_{k-2}}{[\Gamma(\beta+1)]^2}\right)x^{(k+\delta)\alpha}=0 \quad (35)$$

If we define

$$\phi(\delta)=\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\delta(\delta-1)+\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\delta-\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2p^2, \quad (36)$$

then we can write Eq. (35) as

$$\phi(\delta)a_ox^{\delta\alpha}+\phi(\delta+1)a_1x^{(\delta+1)\alpha}+\sum_{k=2}^{\infty}\left(\phi(\delta+k)a_k+\frac{i^{2\alpha}a_{k-2}}{[\Gamma(\beta+1)]^2}\right)x^{(k+\delta)\alpha}=0. \quad (37)$$

For $a_o\neq 0$, we have

$$\phi(\delta)=\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2(\delta^2-p^2)=0,\ \left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\neq 0, \quad (38)$$

which implies that

$$\delta_1=p,\delta_2=-p. \quad (39)$$

case(I): For $p=0$

$$\phi(\delta_1+1)a_1=\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2a_1=0,\left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2\neq 0\Rightarrow a_1=0; \quad (40)$$

$$\phi(\delta_k+1)a_k+\frac{i^{2\alpha}a_{k-2}}{[\Gamma(\beta+1)]^2}=0\Rightarrow a_k=\frac{-i^{2\alpha}}{(k\alpha)^2}a_{k-2},k\geq 2. \quad (41)$$

From the recurrence relation Eq. (41), the odd numbered coefficients vanish, and for the even numbered coefficients we have

$$a_2=\frac{-i^{2\alpha}}{(2\alpha)^2}a_0$$

$$a_4=\frac{-i^{2\alpha}}{(4\alpha)^2}a_2=\frac{-i^{2\alpha}}{(4\alpha)^2}.\frac{-i^{2\alpha}}{(2\alpha)^2}a_0=\frac{i^{4\alpha}}{(2\alpha)^{2(2)}(2!)^2}a_0$$

$$a_6=\frac{-i^{2\alpha}}{(6\alpha)^2}a_4=\frac{-i^{2\alpha}}{(6\alpha)^2}.\frac{i^{4\alpha}}{(2\alpha)^{2(2)}(2!)^2}a_0=\frac{-i^{4\alpha}}{(2\alpha)^{2(3)}(3!)^2}a_0$$

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$$a_{2k}=\frac{(-1)^ki^{2k\alpha}}{(2\alpha)^{2k}(k!)^2}a_0=\frac{i^{2(\alpha+1)k}}{(2\alpha)^{2k}(k!)^2}a_0. \quad (42)$$

Hence, the first solution of the M-truncated fractional modified Bessel equation of zero order is

$$f_1(x) = I_0^\alpha(x) = a_0 \sum_{k=0}^{\infty} \frac{i^{2(\alpha+1)k}}{(k!)^2}\left(\frac{x^\alpha}{2\alpha}\right)^{2k}. \tag{43}$$

Case (II): $P \neq 0$

For $p > 0$, we have

$$\phi(\delta_1 + 1)a_1 = \left[\frac{\alpha}{\Gamma(\beta+1)}\right]^2 (2p+1)a_1 = 0 \Rightarrow a_1 = 0. \tag{44}$$

Equating the coefficient of $x^{(\delta+k)\alpha}$, $k \geq 2$ in Eq. (37) to zero, we get the following recurrence relation

$$a_k = \frac{-i^{2\alpha}}{k(2p+k)\alpha^2} a_{k-2},\, k \geq 2. \tag{45}$$

From $a_1 = 0$ and the recurrence relation above we get

$$a_3 = a_5 = \cdots = 0,$$

and

$$a_2 = \frac{-i^{2\alpha}}{(2\alpha)^2(p+1)} a_0,$$

$$a_4 = \frac{-i^{2\alpha}}{4(2p+4)\alpha^2} a_2 = \frac{-i^{2\alpha}}{4(2p+4)\alpha^2} \cdot \frac{-i^{2\alpha}}{(2\alpha)^2(p+1)} a_0 = \frac{i^{4\alpha}}{(2\alpha)^{2(2)}2!(p+1)(p+2)} a_0,$$

$$a_6 = \frac{-i^{2\alpha}}{6(2p+6)\alpha^2} a_2 = \frac{-i^{2\alpha}}{6(2p+6)\alpha^2} \cdot \frac{i^{4\alpha}}{(2\alpha)^{2(2)}(2!)^2(p+1)(p+2)} a_0 = \frac{-i^{6\alpha}}{(2\alpha)^{2(3)}3!(p+1)(p+2)(p+3)} a_0,$$

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$$a_{2k} = \frac{i^{2(\alpha+1)k}}{(2\alpha)^{2k}k!(p+1)(p+2)\ldots(p+k)} a_0. \tag{46}$$

If we take $a_0 = \frac{1}{(2\alpha)^p\Gamma(p+1)}$, the first solution of the M-truncated fractional modified Bessel differential equation is

$$f_1(x) = I_p^\alpha(x) = \sum_{k=0}^{\infty} \frac{i^{2(\alpha+1)k}(x^\alpha)^{2(p+k)}}{(2\alpha)^{p+2k}(k!)\Gamma(p+1)(p+1)(p+2)\ldots(p+k)} = \sum_{k=0}^{\infty} \frac{i^{2(\alpha+1)k}}{k!\Gamma(k+p+1)}\left(\frac{x^\alpha}{2\alpha}\right)^{p+2k}. \tag{47}$$

3.3 Orthogonality of the FMBFs

In the classical sense, two function f, g are said to be orthogonal on the interval $[a, b]$; if $\int_a^b f(x)g(x)dx = 0$. Understanding the orthogonality relation of the FMBFs is mandatory to compute coefficients of series whose terms include them. These series represent solutions of the fractional differential equations. In the view of M-truncated integral definition Eq. (14), we introduce the following interesting results on orthogonality of FMBFs.

Theorem 3.3.1 If λ and μ are roots of the equation $I_p^\alpha(\xi x) = 0$, then the orthogonality relation of the FMBFs $I_p^\alpha(x)$ over the interval $[0, \xi]$ with respect to the weight function $\omega(x) = x^{2\alpha-1}$ is defined by

$$\int_0^\xi x^{2\alpha-1} I_p^\alpha(\lambda x) I_p^\alpha(\mu x) dx = \frac{\Gamma(\beta+1)\xi^{2\alpha}}{2\alpha i^{2\alpha}} \left(I_p^\alpha(\lambda\xi)\right)^2 \delta_{\lambda\mu}, \alpha \in (0,1], \quad (48)$$

where $\delta_{\lambda\mu}$ is the familiar kronker delta function.

Proof

Since $I_p^\alpha(x)$ is a solution of Eq. (28), it follows that $y = I_p^\alpha(\lambda x)$ satisfies the more general differential equation

$$x^{2\alpha}\, {}_i^M\mathfrak{D}^{\alpha,\beta}\, {}_i^M\mathfrak{D}^{\alpha,\beta} y(x) + \frac{\alpha}{\Gamma(\beta+1)} x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\eta} y(x) + \frac{1}{[\Gamma(\beta+1)]^2}(i^{2\alpha}\lambda^{2\alpha}x^{2\alpha} - p^2)y(x) = 0. \quad (49)$$

It is convenient to reformulate Eq. (49) in the following way:

$$x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\eta} y(x)\right) + \frac{1}{[\Gamma(\beta+1)]^2}(i^{2\alpha}\lambda^{2\alpha}x^{2\alpha} - p^2)y(x) = 0. \quad (50)$$

Consequently, $I_p^\alpha(\lambda x)$ and $I_p^\alpha(\mu x)$ satisfy the following M-truncated fractional differential equations, respectively:

$$x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\lambda x)\right) + \frac{1}{[\Gamma(\beta+1)]^2}(i^{2\alpha}\lambda^{2\alpha}x^{2\alpha} - p^2)I_p^\alpha(\lambda x) = 0, \quad (51)$$

$$x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\mu x)\right) + \frac{1}{[\Gamma(\beta+1)]^2}(i^{2\alpha}\mu^{2\alpha}x^{2\alpha} - p^2)I_p^\alpha(\mu x) = 0. \quad (52)$$

Multiplying Eq. (51) by $x^{-\alpha}I_p^\alpha(\mu x)$ and Eq. (52) by $x^{-\alpha}I_p^\alpha(\lambda x)$ and then subtracting the resulting equations, we get

$$\frac{1}{[\Gamma(\beta+1)]^2}(\lambda^{2\alpha} - \mu^{2\alpha})i^{2\alpha}x^\alpha I_p^\alpha(\lambda x)I_p^\alpha(\mu x) = I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}^{\alpha,\beta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\mu x)\right) - I_p^\alpha(\mu x)\, {}_i^M\mathfrak{D}^{\alpha,\beta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\lambda x)\right). \quad (53)$$

In view of the M-truncated fractional integral formula Eq. (14), we α-integrate this expression over the interval $[0, \xi]$, which gives the following:

$$\frac{1}{\Gamma(\beta+1)}(\lambda^{2\alpha} - \mu^{2\alpha})i^{2\alpha}\int_0^\xi x^{2\alpha-1} I_p^\alpha(\lambda x)I_p^\alpha(\mu x)dx = \int_0^\xi I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}^{\alpha,\eta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\mu x)\right)\frac{dx}{x^{1-\alpha}} - \int_0^\xi I_p^\alpha(\mu x)\, {}_i^M\mathfrak{D}^{\alpha,\beta}\left(x^\alpha\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\lambda x)\right)\frac{dx}{x^{1-\alpha}}. \quad (54)$$

Applying integration by parts on the right-hand side and dividing both sides by the factor $\Gamma(\beta+1)(\lambda^{2\alpha}-\mu^{2\alpha})i^{2\alpha}$, we get

$$\int_0^\xi x^{2\alpha-1} I_p^\alpha(\lambda x) I_p^\alpha(\mu x) dx = \frac{\Gamma(\beta+1)}{(\lambda^{2\alpha}-\mu^{2\alpha})i^{2\alpha}}\left[x^\alpha\left(I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\mu x) - I_p^\alpha(\mu x)\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\lambda x)\right)\right]_0^\xi. \quad (55)$$

Hence, according to the values of λ and μ, we consider the following two cases:

(I) If $\lambda \neq \mu$, the integral term vanishes at the lower limit because $x = 0$, and it also vanishes at the upper limit because $I_p^\alpha(\lambda\xi) = I_p^\alpha(\mu\xi) = 0$. Hence, if $\lambda \neq \mu$, Eq. (55) gives

$$\int_0^\xi x^{2\alpha-1} I_p^\alpha(\lambda x) I_p^\alpha(\mu x) dx = 0. \quad (56)$$

(II) If $\lambda = \mu$, then the resulting integral

$$I = \int_0^\xi x^{2\alpha-1}\left(I_p^\alpha(\lambda x)\right)^2 dx, \quad (57)$$

Creates an interest to look at. To deduce its value, we take the limit of Eq. (55) as $\mu \to \lambda$. As the right- hand side in Eq. (55) approaches the indeterminate form $\frac{0}{0}$ in the limit, we apply L'Hopital's rule as follows:

$$I = \lim_{\mu\to\lambda} \frac{\Gamma(\beta+1)}{(\lambda^{2\alpha}-\mu^{2\alpha})i^{2\alpha}}\left[x^\alpha\left(I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\mu x) - I_p^\alpha(\mu x)\, {}_i^M\mathfrak{D}^{\alpha,\beta} I_p^\alpha(\lambda x)\right)\right]_0^\xi =$$
$$\lim_{\mu\to\lambda} \frac{\Gamma(\beta+1)}{(-2\alpha\mu^{2\alpha})i^{2\alpha}}\left[x^\alpha\left(I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}_\mu^{\alpha,\beta}\, {}_i^M\mathfrak{D}_x^{\alpha,\beta}\, I_p^\alpha(\mu x) - {}_i^M\mathfrak{D}_\mu^{\alpha,\beta}\, I_p^\alpha(\mu x)\, {}_i^M\mathfrak{D}_x^{\alpha,\beta}\, I_p^\alpha(\lambda x)\right)\right]_0^\xi =$$
$$\frac{\Gamma(\beta+1)}{(2\alpha\lambda^{2\alpha})i^{2\alpha}}\left[x^\alpha\left({}_i^M\mathfrak{D}_\lambda^{\alpha,\beta}\, I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}_x^{\alpha,\beta}\, I_p^\alpha(\lambda x) - I_p^\alpha(\lambda x)\, {}_i^M\mathfrak{D}_\lambda^{\alpha,\beta}\, {}_i^M\mathfrak{D}_x^{\alpha,\beta}\, I_p^\alpha(\lambda x)\right)\right]_0^\xi \quad (58)$$

Using the following recurrence relations of MFBFs,

$${}_i^M\mathfrak{D}_x^{\alpha,\beta}\, I_p^\alpha(\lambda x) = \frac{p}{\lambda^\alpha} I_p^\alpha(\lambda x) + i^{2(\alpha+1)} x^\alpha I_{p+1}^\alpha(\lambda x), \quad (59)$$

$${}_i^M\mathfrak{D}_\lambda^{\alpha,\beta}\, I_p^\alpha(\lambda x) = \frac{p}{x^\alpha} I_p^\alpha(\lambda x) + i^{2(\alpha+1)} \lambda^\alpha I_{p+1}^\alpha(\lambda x), \quad (60)$$

it follows that

$$I = \frac{\Gamma(\beta+1)}{2\alpha i^{2\alpha}}\left[\frac{p^2}{\lambda^{2\alpha}}\left(I_p^\alpha(\lambda x)\right)^2 + x^{2\alpha}\left(I_{p+1}^\alpha(\lambda x)\right)^2 + \frac{i^{2(\alpha+1)} x^\alpha p}{\lambda^\alpha} I_p^\alpha(\lambda x) I_{p+1}^\alpha(\lambda x)\right]_0^\xi$$
$$= \frac{\Gamma(\beta+1)\xi^{2\alpha}}{2\alpha i^{2\alpha}}\left(I_{p+1}^\alpha(\lambda\xi)\right)^2. \blacksquare \quad (61)$$

As a special case of theorem 3.3.1, the following result can be easy verified.

Corollary 3.3.1. The FMBFs are orthogonal over the interval [0,1] with respect to the weight function $\omega(x) = x^{2\alpha-1}$ and $\int_0^1 x^{2\alpha-1} I_n^\alpha(\lambda x) I_n^\alpha(\mu x) dx = \frac{\Gamma(\beta+1)}{2\alpha i^{2\alpha}}\left(I_{p+1}^\alpha(\lambda)\right)^2 \delta_{\lambda\mu}$. (62)

Using the orthogonality property Eq. (48), one can easily represent a given function $f(x)$ over the interval $[0, \xi]$ by a series of a Bessel functions such as

$$f(x) = \sum_{i=0}^{\infty} a_i I_p^{\alpha}(\lambda x), 0 < x < \xi, \tag{63}$$

where $I_p^{\alpha}(\lambda\xi) = 0$, $i = 0, 1, 2, 3, \dots$, and a_i are determined by

$$a_i = \frac{2\alpha i^{2\alpha}}{\Gamma(\beta+1)\xi^{2\alpha}\left(I_{p+1}^{\alpha}(\lambda\xi)\right)^2} \int_0^{\xi} x^{2\alpha-1} f(x) I_p^{\alpha}(\lambda x)dx, i = 0, 1, 2, 3, \dots \tag{64}$$

4. Conclusion

Some crucial features of the FMBFs in view of the M-truncated derivative are obtained in this work. The M-truncated fractional modified Bessel differential equation is solved via power series for its first solution, and the orthogonality relation of such functions in the interval $[0, \xi]$ is introduced and analytically proved. The findings of this study are taken as evidence that the results in the sense of the local M-truncated fractional derivative and the results in terms of the classical integer order calculus are consistent. Our results can be extended into applications in future research studies.

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