Continuation of a Parameterized Impulsive Differential Equation to an Internal Nonlocal Cauchy Problem

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Abstract
In this paper, we are concerned with an internal nonlocal Cauchy problem and a parameterize problem of impulsive differential equation. The existence of solutions are proved. The continuations of the parameterize problem of impulsive differential equation and its solution to the internal nonlocal Cauchy problem and its solution will be studied.

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1 Introduction
Impulsive differential equations, by means, differential equations involving impulse effects, are seen as a natural description of observed evolution phenomena of several real world problems. For example, population dynamics [1], physics, Chemistry [2], engineering [3], ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so on. The quantitive investigation of impulsive differential equations began in 1960 with the work of Mil'man and Myshkis [4]. Inrecent years, there have been intensive studies on the qualitative behavior of solutions of impulsive differential equations; see for instance [4, 5, 6, 7, 8, 9]. Basically, impulsive differential equations consist of three components. A continuous-time differential equation, which governs the state of the system impulses, an impulse equation, which model an impulsive jump, defined by a jump function at the instant an impulse occurs and a jump criterion, which define a set of jump events[10]. According to the way in which the moments of the change by jumps are determined, the impulsive differential equations are classified as follows:-

I. Equations with fixed moments of impulse effect (the moments of jump are previously fixed).

II. Equations with unfixed moments of impulse effect (the moments of jump occur when certain space-time relations are satisfied).

And the equations with unfixed moments of impulse effect is complicated, particularly if their solutions considered in an infinite interval [11].There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments.

Consider the internal nonlocal Cauchy problem:

$$x'(t) = f(t, x(t)), \qquad t \in (0, T], \tag{1.1}$$

$$x(\tau) = x_o, \qquad \tau \in (0, T), \tag{1.2}$$

and the parameterized problem of impulsive differential equation

$$x'(t) = f(t, x(t)), \qquad t \in (0, T] \text{ and } t \neq \tau, \tag{1.3}$$

$$x(\tau^-) = \alpha x(\tau^+) = x_o, \qquad \alpha \in (0,1). \tag{1.4}$$

Where $f : [0,T] \times R \to R$ is a given function, $x_o \in R$, $x(\tau^+) = \lim_{h \to o^+} x(\tau + h)$ and $x(\tau^-) = \lim_{h \to o^-} x(\tau + h)$ represent the right and left limits of $x(t)$ at $t = \tau$.

Our aim here is to study the continuation of the problem (1.3)-(1.4) and its solution to the problem (1.1)-(1.2) and its solution, as $\alpha \to 1$.

2 Preliminaries

In this section, we need some basic definitions and properties of impulsive differential equation which are used throughout this paper. By $C[0,T]$ we denote the Banach space of all continuous functions, defined on $[0,T]$ with the norm

$$\| x \|_C = \sup \{ | x(t) | : t \in [0,T] \},$$

set $PC([0,T], R) = \{ x : [0,T] \to R \text{ is continuous everywhere excapt for } t = \tau \text{ at which } x(\tau^-) \text{ and } x(\tau^+) \text{ exist and } x(\tau^-) = x(\tau) \}$

with the norm

$$\| x \|_{PC([0,T],R)} = \sup \{ | x(t) | : t \in [0,T] \}.$$

Definition 2.1

([6, 12]) $x(t)$ is said to be the solution of problem (1.3)-(1.4) if it satisfies the following conditions:

(1) for $(0, +\infty)$, $t \neq \tau$, $x(t)$ is differentiable and $x'(t) = f(t, x(t))$,

(2) $x(t)$ is left continuous in $(0, +\infty)$ and if $t = \tau$, then $x(\tau^+) = \alpha x(\tau^-) = x_o$, $\alpha \neq 1$.

3 Main Results

3.1 Internal nonlocal problem

Now, consider the internal nonlocal problem (1.1)-(1.2).

Definition 3.1

By a solution of problem (1.1)-(1.2), we mean a function $x \in C([0,T],R)$ that satisfies the problem (1.1)-(1.2) itself .

Theorem 3.1

Let $f:[0,T]\times R \to R,$ *is continuous function and satisfies the lipschitz condition*

$$\| f(t,x(t)) - f(t,\bar{x}(t)) \| \le k \| x-\bar{x} \|, \ \forall \ (t,x),(t,\bar{x}) \in [0,T]\times R,$$

with lipschitz constant $k>0$.

If

$$kT < 1 \ . \tag{3.1}$$

Then, the problem (1.1)-(1.2) has a unique solution. This solution can be expressed by the formula

$$x(t) = \begin{cases} x_0 - \int_t^\tau f(s,x(s))\,ds & if \ \ t\in(o,\tau] \\ x_0 + \int_\tau^t f(s,x(s))\,ds & if \ \ t\in(\tau,T]\,. \end{cases} \tag{3.2}$$

Proof

Integrating equation(1.1), we get

$$x(t) = x(0) + \int_0^t f(s,x(s))\,ds, \tag{3.3}$$

let, $t=\tau$ in (3.3), we can deduce that

$$x(\tau) = x(0) + \int_0^\tau f(s,x(s))\,ds,$$

from (1.2), we have

$$x(0) = x_o - \int_0^\tau f(s,x(s))\,ds, \tag{3.4}$$

substituting from (3.4) into (3.3), we have

$$x(t) = x_o + \int_\tau^t f(s,x(s))\,ds. \tag{3.5}$$

Applying the Banach contraction fixed point theorem, we can deduce that the integral equation (3.5) has a unique solution $x \in C[0,T]$. This solution satisfies the problem (1.1)-(1.2) .

Now, let $t\in(0,\tau]$ in (3.5) we obtain

$$x(t) = x_o - \int_t^\tau f(s,x(s))\,ds. \tag{3.6}$$

Also, let $t \in (\tau, T]$ in (3.5), we obtain

$$x(t) = x_o + \int_{\tau}^{t} f(s, x(s))\, ds. \qquad (3.7)$$

Combining (3.6)-(3.7), we obtain the result of the theorem.

3.2 Impulsive differential equation

Now, consider the problem (1.3)-(1.4).

Definition 3.2

By a solution of problem (1.3)-(1.4), we mean a function $x \in PC([0,T], R)$ that satisfies the problem (1.3)-(1.4).

Theorem 3.2

Let the assumptions of theorem (3.1) are satisfied. Then the problem (1.3)-(1.4) has a unique solution.

Proof

Integrating equation (1.3) and using (1.4) we obtain

$$x(t) = \begin{cases} x_o - \int_{t}^{\tau} f(s, x(s))\, ds & \text{if} \quad t \in (0, \tau] \\ \dfrac{x_o}{\alpha} + \int_{\tau}^{t} f(s, x(s))\, ds & \text{if} \quad t \in (\tau, T]. \end{cases} \qquad (3.8)$$

Applying the Banach contraction fixed point theorem, we deduce that there exist a unique solution $x_\alpha \in PC([0,T], R)$ of integral equation (3.8). This solution satisfies the problem (1.3)-(1.4).

3.3 Continuation theorem

Now, we have the following theorem:

Theorem 3.3

If $\alpha \to 1$, then the problems (1.3)-(1.4) and (1.1)-(1.2) are coincide with the same solution.

Proof

Letting $\alpha \to 1$ in (1.4), then the problem (1.3)-(1.4) coincide with the problem (1.1)-(1.2). Let $x(t)$, $x_\alpha(t)$ are given by (3.2) and (3.8) respectively, then

$$\lim_{\alpha \to 1} x_\alpha(t) = x(t), \quad t \in (0, T]. \qquad (3.9)$$

And the two problems (1.1)-(1.2),(1.3)-(1.4) have the same solution.

4 Examples

In this section, we consider some first order impulsive differential equations and the following examples will be helpful to illustrate the main results of this paper.

Example 4.1

Consider the following impulsive differential equation

$$x' = x\cos(t), \quad t \neq \frac{1}{2}, \quad t \in (0, \frac{3}{2}]$$
$$x(\frac{1}{2}^{-}) = \alpha\, x(\frac{1}{2}^{+}) = 1, \quad t = \frac{1}{2}$$

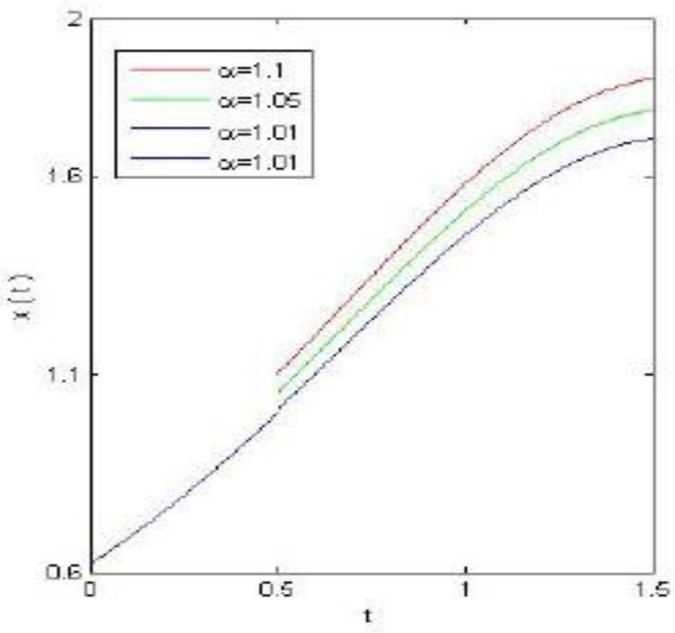

Fig.1 The continuation of solutions of Ex. (4.1)

Example 4.2. *Consider the following impulsive differential equation*

$$x' - t\ \exp(x) = 0,\ \ t \neq \frac{1}{2},\ \ t \in (0,1]$$

$$x(\frac{1}{2}^{-}) = \alpha\, x(\frac{1}{2}^{+}) = \frac{1}{2},\ \ t = \frac{1}{2}$$

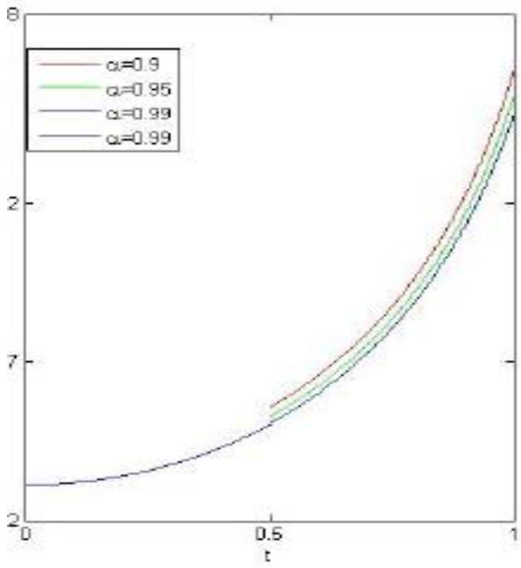

Fig.1 The continuation of solutions of Ex. (4.2)

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