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Cluster-Cluster Potentials for the Beryllium Isotopes

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Abstract

Two types of cluster-cluster potentials are obtained for the beryllium isotopes. The first one of the suggested four-body potentials is an attractive Gaussian interaction and the second is a superposition of repulsive and attractive Gaussian forces. The ground-state wave functions of the beryllium isotopes ^{8}Be, ^{9}Be and ^{10}Be are expanded in series in terms of products of two wave functions: the first represents the set of A-4 nucleons and the second represents the set of the last four nucleons. We use the four-particle fractional parentage coefficients to separate the wave function into a function of (A-4)-nucleons and the other for the remaining alpha particle. In the construction of these wave functions we have used the basis functions of the translation-invariant shell model. The expansion coefficients of the factorizations of these wave functions are products of four-particle orbital and spin-isospin fractional parentage coefficients.

Keywords: Translation invariant shell model, four-particle fractional parentage coefficients, cluster-cluster potentials.

1. Introduction

There are numerous phenomena of nuclear behavior that suggest the clustering of nucleons into groups within a nucleus. The earliest and simplest nuclear model to consider such characteristics is the α-particle model. The fact that four nucleons in a relative $^{11}S_0$ state could strongly interact played an important role in accounting for the binding energy of nuclei. Nuclear ground states would thus be expected to favor such quasi-α particle configurations and also consequently exhibit large spatial symmetry.

Phenomenological cluster-cluster potentials are of interest for applications to α-cluster models [1-5], for the correlation of experimental α-α scattering and for comparison with the results of theoretical studies of the interaction. Most of these studies make use of nucleon-nucleon forces and the resonating group structure method with the α-particles constrained to be in the ground state.

In a previous paper by Doma et al. [5] the authors obtained two Gaussian $\alpha - \alpha$ potentials which are suitable for the calculations of the ground states of the four lithium isotopes, namely: ^{6}Li, ^{7}Li, ^{8}Li and ^{9}Li, by constructing the suitable clusters for these nuclei. One of these potentials is attractive and the other is a superposition of attractive and repulsive interactions. In these calculations they used the basis functions of the translation invariant shell model [6-8] with number of quanta of excitations N up to 6, 7, 8 and 9 for the nuclei ^{6}Li, ^{7}Li, ^{8}Li and ^{9}Li, respectively. In these calculations the authors used the basis functions of the translation invariant shell model [6-8] with number of quanta of excitations N up to 6, 7, 8 and 9 for the nuclei ^{6}Li, ^{7}Li, ^{8}Li and ^{9}Li, respectively.

In the present paper we proceed as in the work of the lithium nuclei [5] to study the structure of the ground-state of the three beryllium isotopes: ^{8}Be, ^{9}Be and ^{10}Be by employing the same two different types of cluster-cluster potentials: the first of which has one attractive Gaussian force and the second consists of a repulsive and an attractive Gaussian forces. The methods of calculating the four-particle orbital and spin-isospin fractional parentage coefficients (f.p.c.) [3] are used to construct the

ground-state nuclear wave functions of the beryllium nuclei in the form of sums of products of two wave functions. The first represents the set of the four-particles and the second represents the set of the A-4 particles added to them the coordinates, the spins, and the isospins of the line joining the center of masses of the two systems. In these calculations we have used the basis functions of the translation invariant shell model [6-8] with number of quanta of excitations N up to 8, 9 and 10 for the nuclei ^{8}Be, ^{9}Be and ^{10}Be, respectively.

2. The Translation Invariant Shell Model Hamiltonian

The internal Hamiltonian operator of the translation invariant shell model (TISM) which describes the mutual motions of the A nucleons relative to the nucleus center of mass has the form [6-8]

$$\mathrm{H}^{(0)} = \frac{1}{\mathrm{A}} \sum_{1=\mathrm{i}<\mathrm{j}}^{\mathrm{A}} \left\{ \frac{\left(\boldsymbol{p}_{\mathrm{i}} - \boldsymbol{p}_{\mathrm{j}}\right)^2}{2\mathrm{m}} + \frac{1}{2}\mathrm{m}\omega^2 \left(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{j}}\right)^2 \right\}. \tag{2.1}$$

A group theoretical method of classifying the states in the TISM was proposed by Kretzschmar [9]. The energy eigenvalues and the total antisymmetric wave functions of a state characterizing such Hamiltonian are given, with the usual notations, by [8]

$$E_N^{(0)} = \left\{ \mathrm{N} + \frac{3}{2}(A-1) \right\} \hbar\omega, \tag{2.2}$$

$$|A\ \Gamma_A> = |A\ N\{\rho\}(\nu)[f]\alpha L M_L S M_S T M_T>. \tag{2.3}$$

In equation (2.3) the number of quanta of excitation N is an irreducible representation (irrep) of the unitary group $U_{3(A-1)}$. The representation $\{\rho\}=\{\rho_1,\rho_2,\rho_3\}$, $\rho_1 \geq \rho_2 \geq \rho_3 \geq 0$ and $\rho_1+\rho_2+\rho_3=N$, is related to Elliot symbol $(\lambda\mu)$ [8,10] by the relations $\lambda=\rho_1-\rho_2$, $\mu=\rho_2-\rho_3$. $\{\rho\}$ is an irrep of the unitary group U_{A-1} and the unitary unimodular subgroup of three dimensions SU_3, at the same time. The representation (ν) is an irrep of the orthogonal group O_{A-1}, $[f]$ is an irrep of the symmetric group S_A, and α is a repetition quantum number of the irreps in the chain of groups. L and M_L stand for the orbital angular momentum and its z-component and are also irreps of the rotational groups SO_3 and SO_2, respectively. S, M_S are the spin, its z-component and T, M_T are the isospin, its z-component which are irreps of the direct product of the groups SU_2 x SU_2. The basis functions (2.3) transform according to the following chain of groups:

$$\begin{array}{ccccccccc}
 & & & & & & SU_4 & \supset & SU_2 \times SU_2 \\
 & & U_4{}^{A} & \supset & U_{4A} & \supset & \times & & \\
\mathfrak{I} & \supset & & & & & U_A & \supset & O_A \supset S_A \\
 & & \times & & & & & & \\
 & & U_{3(A-1)} & \supset & U_{A-1} & \supset & O_{A-1} & \supset & S_A \\
 & & & & & & \times & & \\
 & & & & & & SU_3 & \supset & SO_3 \supset SO_2.
\end{array}$$

Wave functions having definite total angular momentum quantum number J, its z-projection M_J, and isotopic spin T, and its z-projection M_T are constructed from the functions (2.3), in the usual manner, as follows

$$\left|\mathrm{AJM_J TM_T}\right\rangle = \sum_{\mathrm{M_L+M_S=M_J}} \left(\mathrm{LM_L, SM_S}\left|\mathrm{JM_J}\right.\right) \left|\mathrm{AN}\{\rho\}(\nu)[\mathrm{f}]\alpha \mathrm{LM_L SM_S TM_T}\right\rangle, \tag{2.4}$$

where $\{\mathrm{LM_L, SM_S}|\mathrm{JM_J}\}$ are the Clebsch-Gordan coefficients of the rotational group SO_3.

The fractional parentage coefficients (f.p.c.) of separation of four particles are given in terms of the f.p.c. of separation of two particles as follows [3]

$$\begin{aligned}\left\langle \mathrm{A}\Gamma_{\mathrm{A}} \left|\mathrm{A}-4\Gamma_{\mathrm{A-4}},4\Gamma_4\right.\right\rangle &= \sum_{\Gamma_{\mathrm{A-2}},\Gamma_2} \left\langle \mathrm{A}\Gamma_{\mathrm{A}} \left|\mathrm{A}-2\Gamma_{\mathrm{A-2}},2\Gamma_2\right.\right\rangle \\ &\times \left\langle \mathrm{A}-2\Gamma_{\mathrm{A-2}} \left|\mathrm{A}-4\Gamma_{\mathrm{A-4}},2\Gamma_2\right.\right\rangle \left\langle 4\Gamma_4 \left|2\Gamma_2,2\Gamma_2\right.\right\rangle \\ &\times \mathrm{U}(\mathrm{L_{A-4}L_2LL_2; L_{A-2}L_4})\Lambda_{[\mathrm{f}]}.\end{aligned} \tag{2.5}$$

The first three factors on the right-hand side of equation (2.5) are two-particles f.p.c. for the sets of A, A-2, and 4-particles, respectively. U is the Racah coefficient and $\Lambda_{[\mathrm{f}]}$ is defined by

$$\Lambda_{[\mathrm{f}]} = \left\{ 6 \frac{\tau_{\mathrm{A,A-3}}\tau_{\mathrm{A-1,A-3}}\tau_{\mathrm{A-1,A-2}}\tau_{\mathrm{A,A-2}}}{(1+\tau_{\mathrm{A,A-3}})(1+\tau_{\mathrm{A-1,A-3}})(1+\tau_{\mathrm{A-1,A-2}})(1+\tau_{\mathrm{A,A-2}})} \right\}^{1/2}, \tag{2.6}$$

where $\tau_{\mathrm{i,j}}$ is the axial distance between the numbers i and j in the Young tableau. Γ_{A} is the set of all orbital, spin, and isotopic spin quantum numbers of the system consisting of A nucleons and similarly for the other systems.

3. The Cluster-Cluster Potentials

For our potentials, we have used two types of interactions having Gaussian-radial dependence. The first consists of one attractive term of the form [5]

$$\mathrm{V}^{(1)}_{\alpha\alpha} = \mathrm{V_{a1}} \exp\left\{-\mathrm{R}^2/\mathrm{R}^2_{\mathrm{a1}}\right\}, \tag{3.1}$$

where $\mathrm{V_{a1}}$ and $\mathrm{R_{a1}}$ are the depth and range parameters of the interaction, respectively. The second potential consists of an attractive and repulsive terms of the form [5]

$$\mathrm{V}^{(2)}_{\alpha\alpha} = \mathrm{V_{a2}} \exp\left(-\mathrm{R}^2/\mathrm{R}^2_{\mathrm{a2}}\right) + \mathrm{V_{r2}} \exp\left(-\mathrm{R}^2/\mathrm{R}^2_{\mathrm{r2}}\right), \tag{3.2}$$

where $\mathrm{V_{a2}}$ and $\mathrm{V_{r2}}$ are the depth parameters of the attractive, and the repulsive terms, respectively. $\mathrm{R_{a2}}$ and $\mathrm{R_{r2}}$ are the corresponding range parameters, respectively. In equations (3.1), and (3.2) $\boldsymbol{R}$ is the inter-alpha distance defined by

$$\boldsymbol{R} = \left(\boldsymbol{R}_{\alpha}^{(1)} - \boldsymbol{R}_{\alpha}^{(2)}\right)/\sqrt{2}. \tag{3.3}$$

Here $\boldsymbol{R}_{\alpha}^{(1)}$ and $\boldsymbol{R}_{\alpha}^{(2)}$ are the radius vectors of the centers of mass of the two clusters.

4. The Four-Particle Matrix Elements

The four-particle matrix elements of the potentials $V_{\alpha\alpha}^{(1)}$, given by equation (3.1), and hence of $V_{\alpha\alpha}^{(2)}$ of equ. (3.2), with respect to the TISM basis functions, equ. (2.3), are given by [5]

$$\begin{aligned}
&\left\langle A\Gamma_A \left| V_{\alpha\alpha}^{(1)} \right| A\Gamma_A^{/} \right\rangle = \sum_{\Gamma_{A-4},\Gamma_4,\Gamma_4^{/}} \frac{A(A-1)(A-2)(A-3)}{4!} \\
&\times \left\langle A\Gamma_A \left| A-4\Gamma_{A-4},4\Gamma_A \right\rangle \left\langle A\Gamma_A^{/} \right| A-4\Gamma_{A-4},4\Gamma_4^{/} \right\rangle \\
&\times \begin{Bmatrix} L_{A-4} & S_{A-4} & J_{A-4} \\ L_A & 0 & J_4 \\ L & S & J \end{Bmatrix} \begin{Bmatrix} L_{A-4} & S_{A-4}^{/} & J_{A-4} \\ L_4^{/} & 0 & J_4 \\ L^{/} & S^{/} & J \end{Bmatrix} \\
&\times V_{a1} \left\langle A-4\Gamma_{A-4},4\Gamma_4 \left| \exp\left(-R^2/R_{a1}^2\right) \right| A-4\Gamma_{A-4},4\Gamma_4^{/} \right\rangle
\end{aligned} \tag{4.1}$$

In equation (4.1) $\begin{Bmatrix} L_1 & S_1 & J_1 \\ L_2 & S_2 & J_2 \\ L_{12} & S_{12} & J_{12} \end{Bmatrix}$ are normalized 9-j symbols.

Accordingly, the Hamiltonian matrices H which are the sum of the TISM-Hamiltonian $H^{(0)}$, given by equation (2.1), and the perturbed ones with elements given by equation (4.1) can be constructed as functions of the oscillator parameter $\hbar\omega$, equation (2.2), and the potential parameters. Diagonalizing these matrices with respect to the oscillator parameter $\hbar\omega$, which is allowed to vary in a wide range of values $8 \le \hbar\omega \le 30$ MeV in order to obtain the best ground-state energy eigenvalue for each nucleus, the ground-state energy eigenvalues and eigenfunctions are then obtained for each nucleus.

5. Results and Conclusions

The attractive depth parameter V_{a1} of the first potential is varied in the range $-50.0 \le V_{a1} \le -10.0$ MeV with a step 1.0 MeV, and the corresponding range parameter R_{a1} is varied in the range $1.5 \le R_{a1} \le 10.0$ fm with a step 0.0001 fm, in order to obtain the best values of the binding energies of the three nuclei ^{8}Be, ^{9}Be and ^{10}Be. The attractive depth parameter of the second potential V_{a2} is allowed to vary in the same energy range as for V_{a1} with step 1.0 MeV. The corresponding range parameter R_{a2} is allowed to vary in the same range as for R_{a1} with a step 0.0001 fm. The repulsive depth parameter V_{r2} is varied in the range $1.0 \le V_{r2} \le 50$ MeV with a step 1.0

MeV. The corresponding range parameter R_{r2} is varied in the range $2.0 \leq R_{r2} \leq 7.5$ fm with a step 0.1 fm.

In Figure-1 we present the variation of the range parameter R_r with respect to the depth parameter V_a for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be. In Fig. 2 we present the variation of the range parameter R_a with respect to the depth parameter V_r for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be. Fig. 3 presents the variation of the range parameter R_r with respect to the depth parameter V_r for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be. In Fig. 4 we present the variation of the range parameter R_a with respect to the depth parameter V_a for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be. Also, in Fig. 5 we present the variation of the range parameter R_r with respect to the depth parameter V_r for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be. Fig. 6 shows the variation of the range parameter R_r with respect to the range parameter R_a for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be. Finally, in Fig. 7 we present the variation of the range parameter R_r with respect to the depth parameter V_a for the three isotopes ^{8}Be, ^{9}Be and ^{10}Be.

As seen from the figures it is always possible to find a potential of the form given by either of equation (3.1) or (3.2), in the considered range of values of the potential parameters, which gives rise to good agreement between the calculated value of the binding energy of one of the beryllium isotopes and the corresponding experimental value.

Indeed, each cluster-cluster potential is a collection of nucleon-nucleon Gaussian potentials arranged in some way in order to average the effects of their mutual interactions.

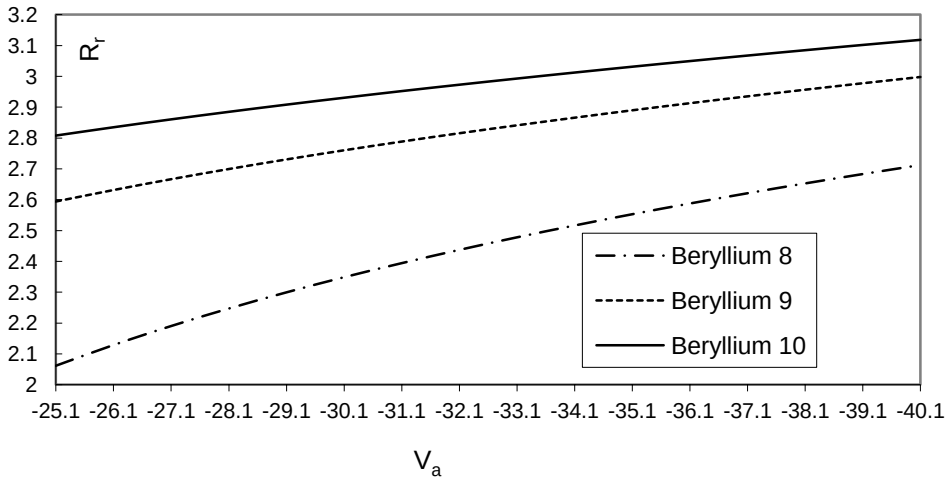

Fig. 1 Variation of the range parameter R_r with respect to the depth parameter V_a.

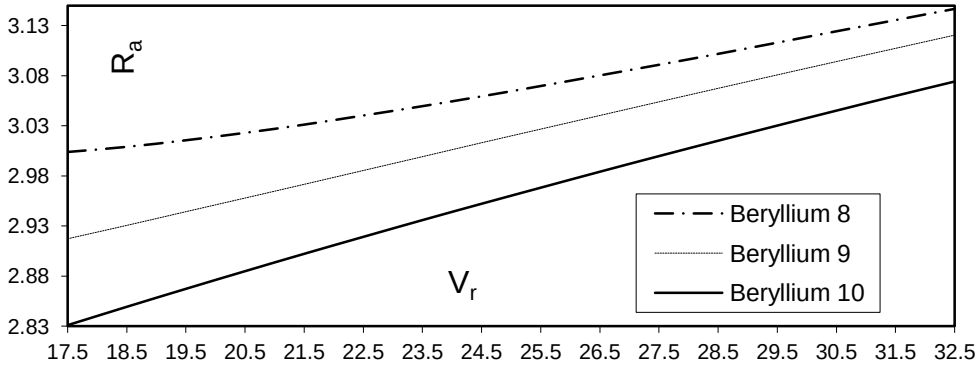

Fig. 2 Variation of the range parameter R_a with respect to the depth parameter V_r.

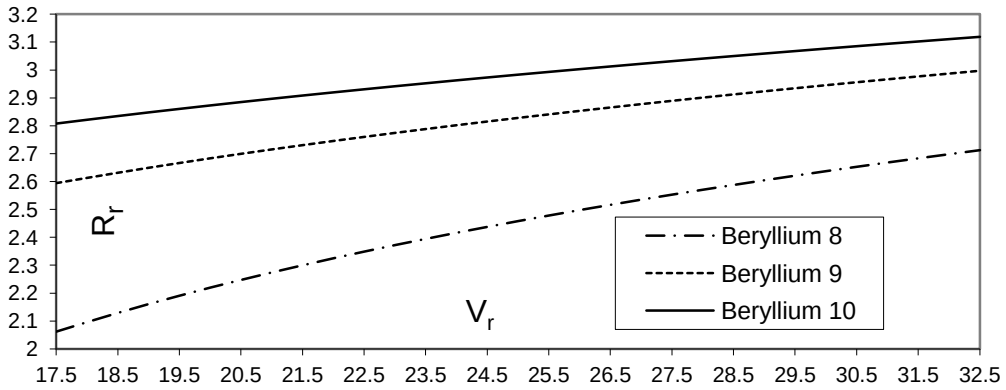

Fig. 3 Variation of the range parameter R_r with respect to the depth parameter V_r.

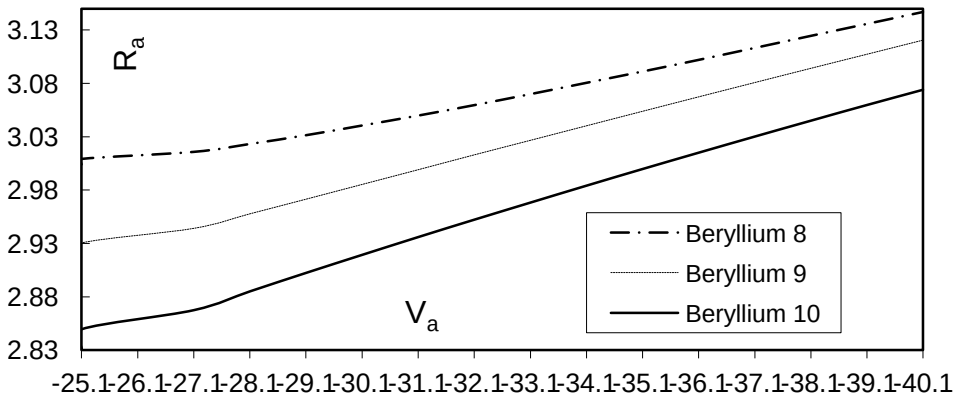

Fig. 4 Variation of the range parameter R_a with respect to the depth parameter V_a.

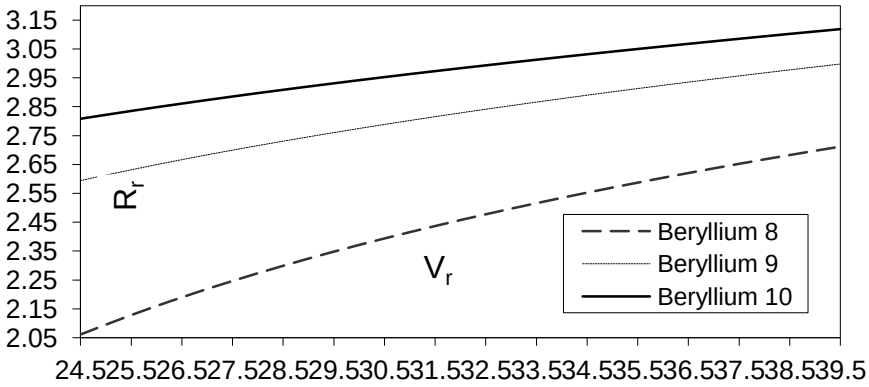

Fig. 5 Variation of the range parameter R_r with respect to the depth parameter V_r.

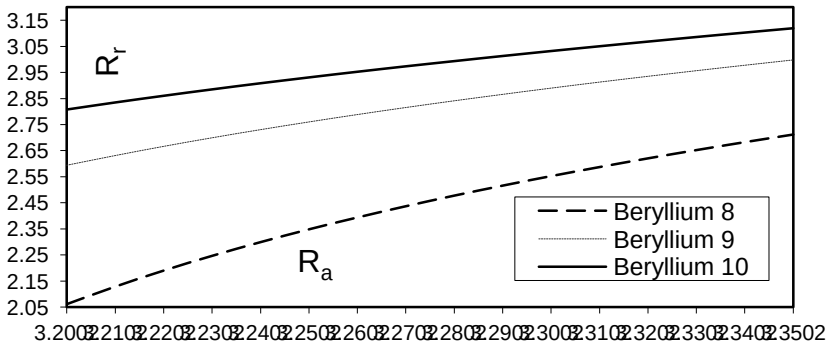

Fig. 6 Vriation of the range parameter R_r with respect to the range parameter R_a.

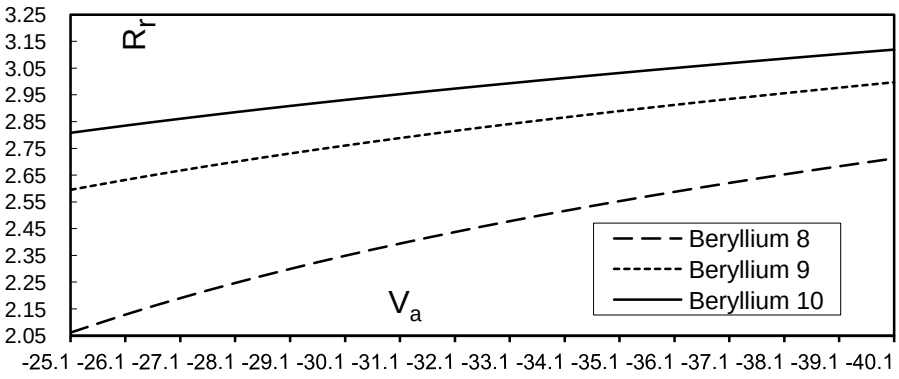

Fig. 7 Variation of the range parameter R_r with respect to the depth parameter V_a

The calculated ground-state nuclear wave function of the nucleus ^{8}Be is a superposition of two S-states with symmetry type [44] which admits the possibility of forming α + α structure of this nucleus. The ground-state of the ^{8}Be nucleus is also a superposition of P, and D-states of symmetry type [431] together with a D-state of symmetry type [422], which admit the possibility of forming the α + t + p, α + ^{3}He + n structures of this nucleus. Other structures are available for this nucleus with smaller weights. The main structure in the ground-state nuclear wave function of the nucleus ^{9}Be is of the form $\alpha + \alpha + n$. Among the other available structures for this nucleus is the one which is a superposition of P and D-states of symmetry type [432], that admits the possibility of forming α + t + d structure of this nucleus. Structure of the form α + ^{3}He + n+n is also available with a smaller weight. Finally, the main structure in the ground-state nuclear wave function of ^{10}Be is of the symmetry type [442] which admits the possibility of forming an $\alpha + \alpha + n + n$-structure for this state.

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