Perturbation Treatment for a Rectangular Membrane Subject to a Restorative Force

S. B. Doma[1], I. H. El-Sirafy[2] and A. H. El-Sharif[3]

[1] Basic Sciences Department, Faculty of Information Technology and Computer Sciences, Sinai University, El-Arish, North Sinai, Egypt.
E-mail address: sbdoma@yahoo.com
[2] Mathematics Department, Faculty of Science, Alexandria University, Alexandria, Egypt.
E-mail address: isirafy@yahoo.com
[3] Mathematics Department, Faculty of Science, Garyounis University, Benghazi, Libya.

Abstract
We have investigated the motion of a stretched elastic rectangular membrane which is subjected to a restorative force proportional to the velocity. The perturbation method is applied to obtain the solutions of the problem in the presence of the restorative force in terms of those in the absence of the perturbation. Furthermore, the roles of the initial displacement and initial velocity are investigated and the numerical solutions are then given by using the program Mathematica. Moreover, the nodal lines of the vibrations are also given.

Keywords: Hyperbolic partial differential equations, vibrations of rectangular membrane, perturbation method, nodal lines.

1. Introduction
Hyperbolic partial differential equations involve second derivatives of opposite sign, such as the wave equation describing the vibrations of a stretched string. Hyperbolic partial differential equations are very essential in engineering and theoretical physics problems. One of the famous problems of their applicability in theoretical physics is the solution of the motion of a relativistic quantum mechanical particle in an electromagnetic field. The problems of vibrating rectangular or circular membrane are also very interesting especially when the membrane is subject to a restorative force.

The vibrating membrane problem can be used as a rather appropriate example to demonstrate the power of computer algebra systems like Axiom, Maple, Mathematica, Derive, etc. [1].

Different methods have been applied for the investigation of vibrating membranes. The differential quadrature method was applied for frequency analysis of rectangular and circular membranes [2,3]. Accordingly, some important studies concerning analysis of membranes have been carried out [4,5]. Furthermore, free vibration analysis of plates and shells has been also investigated [6,7].

The simplest method for solving the problem of a vibrating rectangular membrane is given, as usual [8], by separating the variables. In the presence of a restorative force, that is proportional to the velocity, the perturbation expansions [9] for eigenvalues and eigenfunctions are also of particular interest. Based upon the known solutions of the problem in the absence of the restorative force one can then derive the solutions of the problem in the presence of the external force in the form of a power series in terms of those solutions.

In a previous paper [10] we have investigated the problem of the vibration of a circular membrane which is subjected to a restorative force, proportional to the velocity, by applying the perturbation method. The displacement at any point (r, θ) on the membrane and any instant of time t has been obtained numerically by applying the program Mathematica. The obtained results showed that the second-order of the approximations produce results in excellent agreement with those obtained by using the method of separation of variables.

In the present paper, we have applied the perturbation method [9] to solve the differential equation, which represents the motion of a stretched elastic rectangular membrane that is subjected to a restorative force proportional to the velocity. The membrane is assumed to be homogeneous, perfectly flexible and offers no resistance to bending. Moreover, in order to simplify the problem, the deflection of the membrane during the motion is supposed to be small compared to the size of the membrane. The displacement of the membrane at any given point (x, y) and instant of time t is given in terms of the solutions of the problem in the absence of the restorative force and the numerical solutions are then given by using the program Mathematica. The roles of the initial displacement and initial velocity are also investigated. Moreover, the nodal lines are also given.

2. Formulation of the Problem

2.1 The Differential equation

The differential equation which governs the motion of the vibrating rectangular membrane is given by [8]

$$\frac{\partial^2 u_0}{\partial t^2} = c^2 \left\{\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right\}, \tag{2.1}$$

where c is a constant, which is given in terms of the tension per unit lengths and the density ρ by the relation

$$c = \sqrt{\frac{T}{\rho}}, \text{ft/sec.} \tag{2.2}$$

The constant c has the dimension of velocity, as expected. The solutions of equation (2.1) are well known when the boundary and initial conditions are stated [8].

When the membrane is subjected to a restorative force that is proportional to the velocity the new differential equation is now given by

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = c^2 \left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\}, \tag{2.3}$$

where k is the proportionality constant.

2.2 The Boundary Conditions

We take u to be zero on the boundary of the membrane, so that

$$u = 0, 0 \le x \le a \text{ and } 0 \le y \le b, \tag{2.4}$$

for all $t \ge 0$.

2.3 The Initial Conditions

The initial displacement is taken, as in practical applications, as follows

$$u(x, y, 0) = f(x, y), \tag{2.5}$$

where $f(x, y)$ is assumed to be a continuous function.

The initial velocity is

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y), \tag{2.6}$$

where $g(x, y)$ is also a continuous function.

3. The Method of Separation of Variables

By separating the variables, one can easily obtain the following solutions for equation (2.1) under the stated above boundary and initial conditions:

$$u_0(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\begin{bmatrix}\{A_{m,n}cos(\lambda_{m,n}t) + B_{m,n}sin(\lambda_{m,n}t)\} \\ \times\, sin\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right)\end{bmatrix}, \tag{3.1}$$

where

$$A_{m,n} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)sin\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right)dxdy, \tag{3.2}$$

$$B_{m,n} = \frac{4}{ab\lambda_{m,n}}\int_0^a\int_0^b g(x,y)sin\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right)dxdy, \tag{3.3}$$

and

$$\lambda_{m,n} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}, \tag{3.4}$$

$m = 1,2,\ldots$ and $n = 1,2,\ldots$.

Furthermore, by separating the variables one obtains the following solutions for equation (2.3)

$$u(x,y,t) = e^{-kt/2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\mathrm{A}_{m,n}\cos(\mu_{m,n}t) + \mathrm{D}_{m,n}\sin(\mu_{m,n}t)\}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{3.5}$$

where $A_{m,n}$ is given by (3.2) and

$$D_{m,n} = \frac{4}{ab\mu_{m,n}}\int_0^a\int_0^b\left\{g(x,y)+\frac{c\pi}{2}f(x,y)\right\}sin\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right)dxdy. \tag{3.6}$$

In equations (3.5) and (3.6) the values $\mu_{m,n}$ are given by

$$\mu_{m,n} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}-\frac{k^2}{4c^2\pi^2}}, \tag{3.7}$$

where $m = 1,2,\ldots$ and $n = 1,2,\ldots$.

The values $\mu_{m,n}$ are called eigenvalues of the rectangular membrane and the functions

$$u_{m,n}(x,y,t) = e^{-kt/2}\{\mathrm{A}_{m,n}\cos(\mu_{m,n}t) + \mathrm{D}_{m,n}\sin(\mu_{m,n}t)\}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{3.8}$$

are called eigenfunctions of the rectangular membrane.

4. The Perturbation Method of Solution

We have now to determine by how much the solutions of equation (2.1), under the boundary and initial conditions stated above, have been changed on account of the presence of the disturbing factor $k\frac{\partial u}{\partial t}$, since it is assumed to be small compared to the other terms. The change is known as a perturbation.

To apply the perturbation method [9] to equation (2.3) we try first to find solutions of the form

$$u = ve^{\frac{-kt}{2}}. \tag{4.1}$$

Equation (2.3), then, becomes

$$c^2(v_{xx} + v_{yy}) = v_{tt} - \frac{k^2}{4} v \tag{4.2}$$

with the initial conditions

$$v(x,y,0) = f(x,y),$$
$$v_t|_{t=0} = g(x,y) + \frac{k}{2} f(x,y) = h(x,y)$$

To apply the perturbation method to equation (4.2) we rewrite it in the form

$$c^2(v_{xx} + v_{yy}) = v_{tt} - \alpha \frac{k^2}{4} v, \tag{4.3}$$

where α is a parameter introduced to know the order of the approximation and takes the value 1 in the final result. Accordingly, the equation representing the case where there is no external restorative force is obtained by putting $\alpha = 0$ in (4.3)

$$c^2\left(v_{xx}^{(0)} + v_{yy}^{(0)}\right) = v_{tt}^{(0)}, \tag{4.4}$$

where $v^{(0)}$ is the corresponding solution in this case.

The second step now is to use the solution $v^{(0)}(x,y,t)$ of equation (4.4) to derive solutions of equation (4.3), satisfying the boundary and initial conditions stated above in the following manner. Assume that the solutions of equations (4.3), $v(x,y,t)$, are expanded in series in powers of α, such that

$$v = v^{(0)} + \alpha v^{(1)} + \alpha^2 v^{(2)} + \cdots . \tag{4.5}$$

Thus, in the presence of the perturbation we have for the zero's order of the approximation the same equation (4.4), as expected, with the boundary and initial conditions given by

$$v^{(0)}(x,y,0) = f(x,y), \tag{4.6}$$

$$v^{(0)}{}_t|_{t=0} = h(x,y). \tag{4.7}$$

For any order of the approximation, we have the system of inhomogeneous wave equations

$$c^2\left(v_{xx}^{(j)} + v_{yy}^{(j)}\right) = v_{tt}^{(j)} - \frac{k^2}{4} v^{(j-1)}, \qquad j = 1,2,3,..., \tag{4.8}$$

with the initial conditions

$$v^{(j)}(x,y,0) = 0, \tag{4.9}$$
$$v^{(j)}{}_t|_{t=0} = 0; \qquad \text{j} = 1, 2, 3, \ldots . \tag{4.10}$$

The solutions of equations (4.4) – (4.10) are given by

$$v^{(j)}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{m,n}^{(j)}(t) \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right), \quad \text{j} = 0, 1, 2, \ldots, \tag{4.11}$$

where

$$w_{m,n}^{(0)}(t) = A_{m,n}\cos(\lambda_{m,n} t) + E_{m,n}\sin(\lambda_{m,n} t). \tag{4.12}$$

In equation (4.12) $A_{m,n}$ is given as before by (3.2) and

$$E_{m,n} = \frac{4}{abc\lambda_{m,n}} \int_{y=o}^{b} \int_{x=0}^{a} h(x,y) \sin\left(\frac{\pi m x}{a}\right) sin\left(\frac{\pi n y}{b}\right) dxdy, \tag{4.13}$$

$$\lambda_{m,n} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\,. \tag{4.14}$$

From (4.8) and (4.11) we get

$$\frac{d^2 w_{m,n}^{(j)}(t)}{dt^2} + \left(\lambda_{m,n}\right)^2 w_{m,n}^{(j)}(t) = \frac{K^2}{4} w_{m,n}^{(j-1)}(t)\,, \quad j = 1,2,3, \dots . \tag{4.15}$$

The solutions of (4.15) can be written in the form:

$$w_{m,n}^{(j)}(t) = \frac{K^2}{4\lambda_{m,n}} \int_0^t w_{m,n}^{(j-1)}(\tau) \sin\{\lambda_{m,n}(t-\tau)\} d\tau\,, \qquad j = 1, 2, 3, \dots . \tag{4.16}$$

Hence, the solutions of (2.3) are finally given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n}(x,y,t) =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} e^{-\frac{kt}{2}} w_{m,n}^{(j)}(t)\alpha^j \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right). \tag{4.17}$$

From (4.12) and (4.16), we have

$$w_{m,n}^{(1)}(t) = \frac{k^2}{8\lambda_{m,n}}\left[A_{m,n} t \sin\left(\lambda_{m,n}t\right) + E_{m,n}\left(\frac{\sin(\lambda_{mn}t)}{\lambda_{m,n}} - t\cos\left(\lambda_{m,n}t\right)\right)\right] \tag{4.18}$$

and

$$\frac{128\left(\lambda_{m,n}\right)^2}{k^4} \times w_{m,n}^{(2)}(t) = A_{m,n}\left\{\frac{t\sin\left(\lambda_{m,n}t\right)}{\lambda_{m,n}} - t^2\cos\left(\lambda_{m,n}t\right)\right\}$$

$$+E_{m,n}\left\{\begin{array}{c}\frac{3}{\left(\lambda_{m,n}\right)^2}\sin\left(\lambda_{m,n}t\right) - \frac{3t}{\lambda_{m,n}} cos\left(\lambda_{m,n}t\right) - t^2 sin\left(\lambda_{m,n}t\right) \\ +\frac{3}{\left(\lambda_{m,n}\right)^2}\left\{sin\left(\lambda_{m,n}t\right)\right\}^3 - \frac{6sin^2(\lambda_{m,n}t)cos(\lambda_{m,n}t)}{\lambda_{m,n}}\end{array}\right\}. \tag{4.19}$$

Accordingly, $v^{(1)}(x,y,t)$ and $v^{(2)}(x,y,t)$ are calculated. From which the functions v and, hence, u are calculated.

5. Results and Conclusion

We have applied the perturbation method to the second-order of the approximation, which is sufficiently enough since the convergence of the solutions for our choice of initial functions and parameters is good.

In the numerical computations we have considered a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft, the constant tension is 12.5 lb/ft, the density is 2.5 slugs/ft^2, as for light rubber. Moreover, the constants k and c are given by

$$k = c\pi/10 \quad \text{and } c^2 = \frac{16}{5\pi^2}\ \text{ft}^2/\text{sec}^2.$$

For the initial displacement we have tested several functions among which the best for producing remarkable vibrations which present the problem very well are

$$f(x,y) = xy(a^2 - x^2)(b^2 - y^2), \tag{5.1}$$
$$f(x,y) = d(ax - x^2)(by - y^2), \tag{5.2}$$

where d is a constant. All of the tested functions are positive continuous function of x and y in the intervals $x \in [0,a]$ and $y \in [0,b]$.

For the initial velocity we also have

$$g(x,y) = 0, \tag{5.3}$$
$$g(x,y) = x + y, \tag{5.4}$$
$$g(x,y) = x - y. \tag{5.5}$$

It is very interesting to note that, depending on a and b several functions $u_{m,n}$, (3.8) or (4.17), may correspond to the same eigenvalue. Physically this means that there may exist vibrations having the same frequency but entirely different nodal lines (curves of points on the membrane that do not move).

In our model the eigenvalues are given by

$$\frac{20\mu_{m,n}}{\pi c} = \sqrt{\mathrm{N}} = \sqrt{25m^2 + 100n^2 - 1}, \quad m = 1, 2, 3, \ldots \text{ and } n = 1, 2, 3, \ldots . \tag{5.6}$$

Accordingly, different functions $u_{m,n}$ may correspond to the same value of $\mu_{m,n}$. For example, we have N = 1999, for m = 4 and n = 4, and also for m = 8 and n = 2. Hence,

$$\mu_{4,4} = \mu_{8,2} = \frac{c\pi}{20}\sqrt{1999}.$$

But for $m = 4$ and $n = 4$, the corresponding function is

$$u_{4,4} = e^{-kt/2}\left\{A_{4,4}cos\left(\frac{c\pi}{20}\sqrt{1999}\mathrm{t}\right) + D_{4,4}sin\left(\frac{c\pi}{20}\sqrt{1999}\mathrm{t}\right)\right\} sin(\pi x)sin(2\pi y). \tag{5.7}$$

And for m = 8 and n = 2, the corresponding function is

$$u_{8,2} = e^{-kt/2}\left\{A_{8,2}cos\left(\frac{c\pi}{20}\sqrt{1999}\mathrm{t}\right) + D_{8,2}sin\left(\frac{c\pi}{20}\sqrt{1999}\mathrm{t}\right)\right\} sin(2\pi x)sin(\pi y). \tag{5.8}$$

These two functions are certainly different and have the nodal lines $x = 1, x = 2, x = 3$, and $y = \frac{1}{2}, y = 1, y = \frac{3}{2}$ in the first case and $x = \frac{1}{2}, x = 1, x = \frac{3}{2}, x = 2, x = \frac{5}{2}, x = 3, x = \frac{7}{2}$ and $y = 1$, in the second case.

For the sake of illustrations we present the vibrations obtained by using the initial displacement given by (5.1) and the initial velocity given by (5.4). In Figures 1-30 we present the displacements $u(x,y,t)$, in ft, for $t = 0, 1, 2, \ldots$, and 29, respectively.

It is seen from the figures that after 5 seconds the displacement is approximately equals to zero. The displacement of the membrane starts to be negative after 6 seconds.

After 13 seconds, the displacement starts to be positive again but with too small values. After 27 seconds the membrane approximately stopped, so that the considered damping force acts for only 27 seconds.

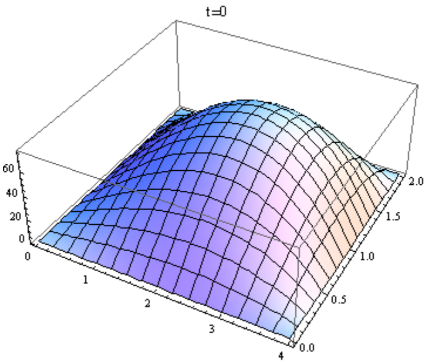

Fig. 1 Displacement for $t = 0$

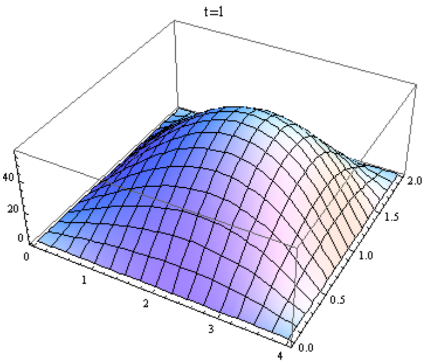

Fig. 2 Displacement for $t = 1$

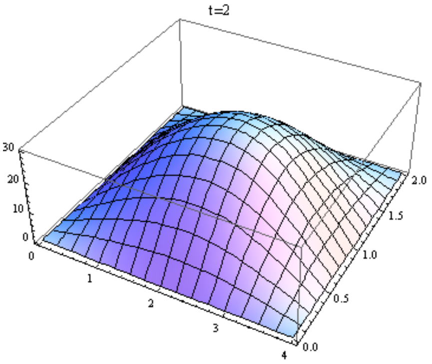

Fig. 3 Displacement for $t = 3$

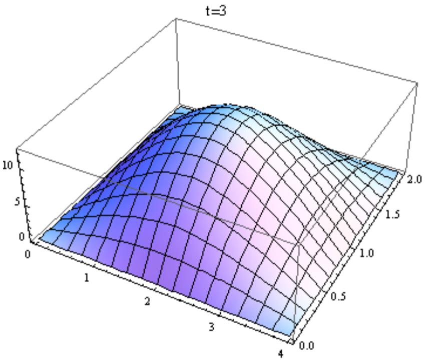

Fig. 4 Displacement for $t = 4$

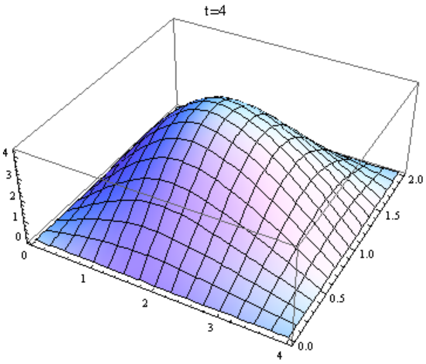

Fig. 5 Displacement for $t = 4$

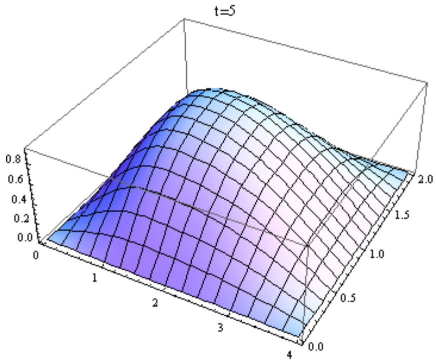

Fig. 6 Displacement for $t = 5$

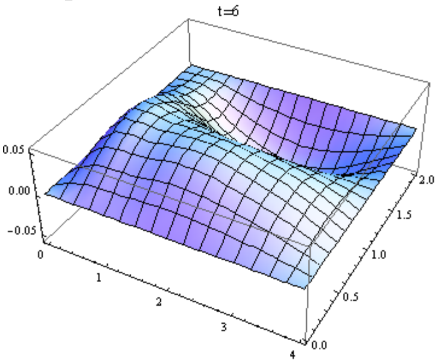

Fig. 7 Displacement for $t = 6$

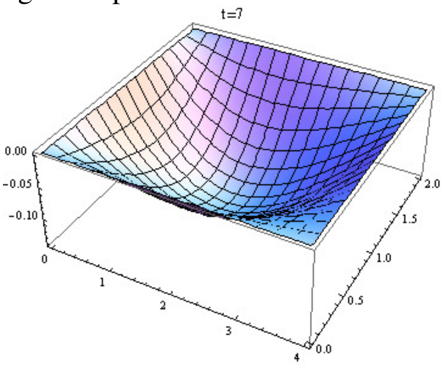

Fig. 8 Displacement for $t = 7$

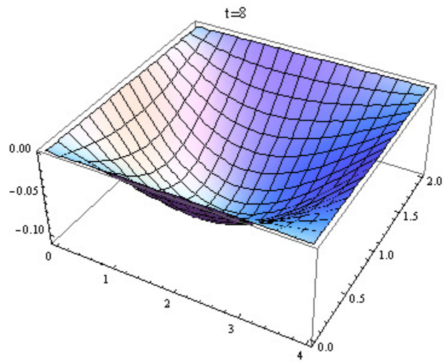

Fig. 9 Displacement for $t = 8$

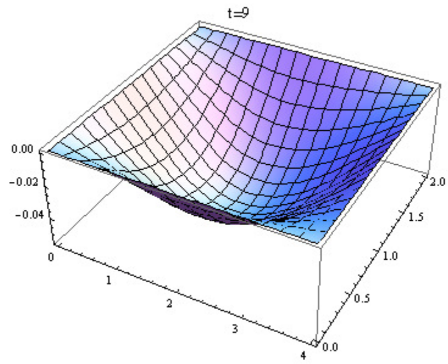

Fig. 10 Displacement for $t = 9$

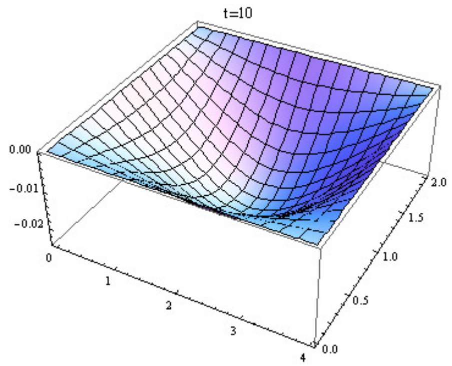

Fig. 11 Displacement for $t = 10$

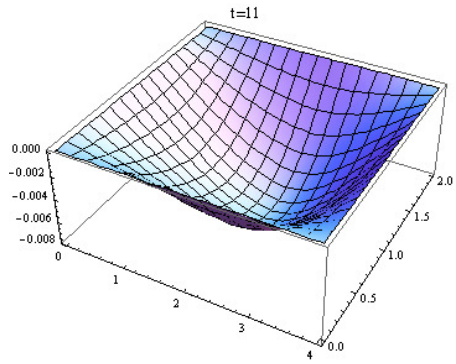

Fig. 12 Displacement for $t = 12$

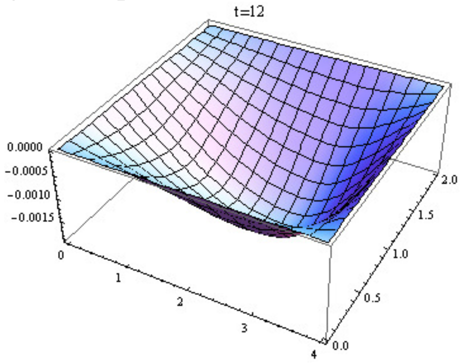

Fig. 13 Displacement for $t = 12$

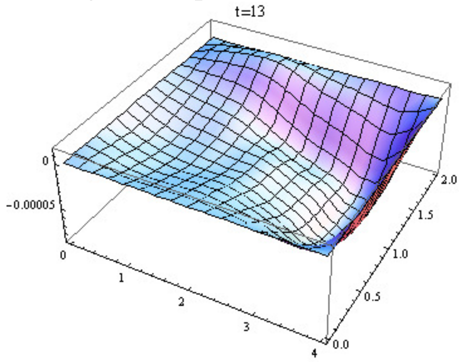

Fig. 14 Displacement for $t = 13$

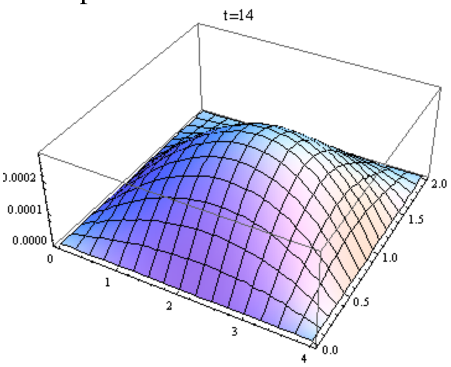

Fig. 15 Displacement for $t = 14$

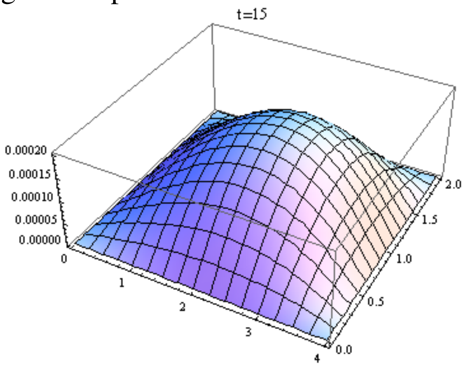

Fig. 16 Displacement for $t = 15$

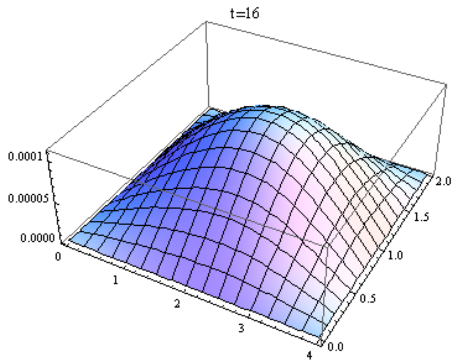

Fig. 17 Displacement for $t = 16$

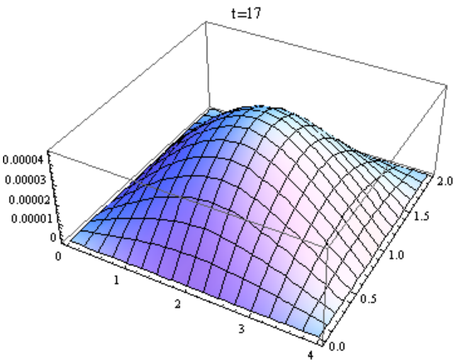

Fig. 18 Displacement for $t = 17$

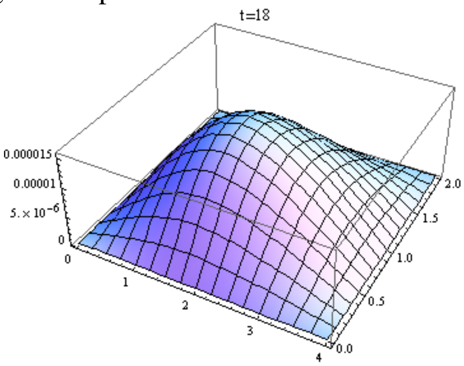

Fig. 19 Displacement for $t = 18$

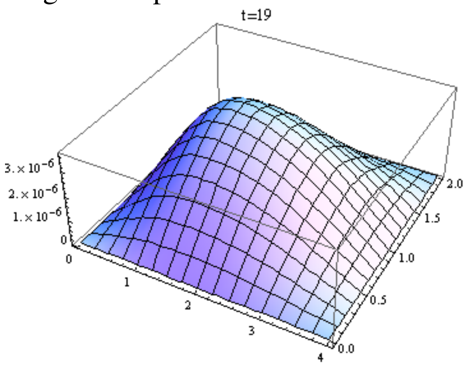

Fig. 20 Displacement for $t = 19$

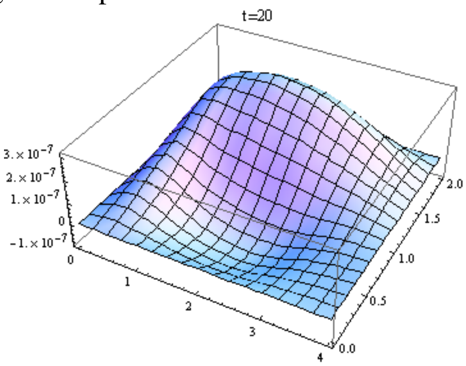

Fig. 21 Displacement for $t = 20$

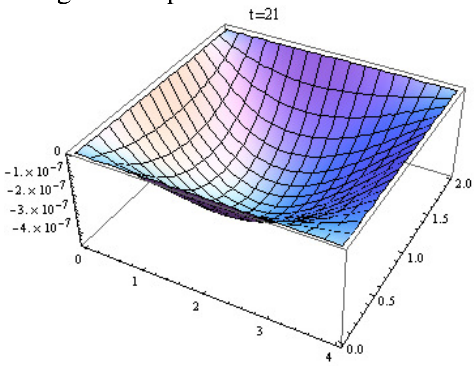

Fig. 22 Displacement for $t = 21$

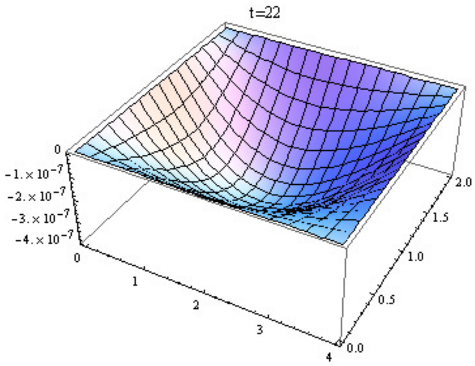

Fig. 23 Displacement for $t = 22$

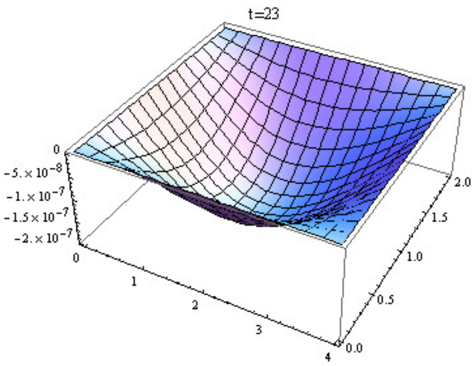

Fig. 24 Displacement for $t = 23$

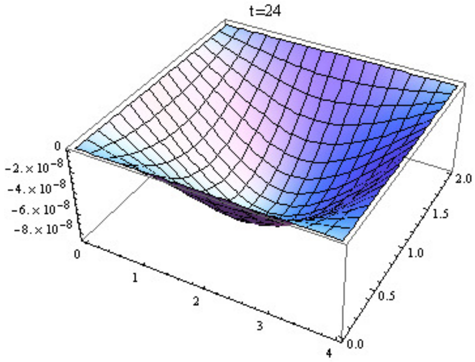

Fig. 25 Displacement for $t = 24$

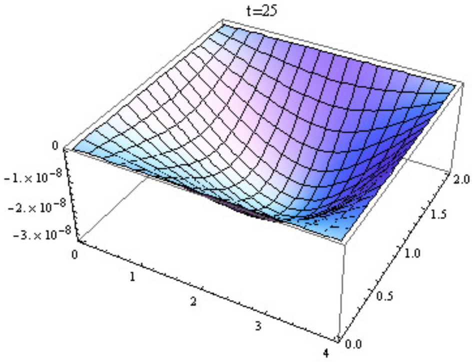

Fig. 26 Displacement for $t = 25$

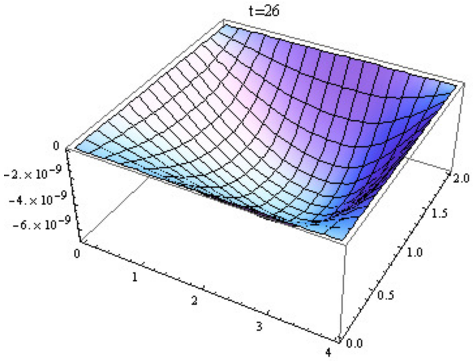

Fig. 27 Displacement for $t = 26$

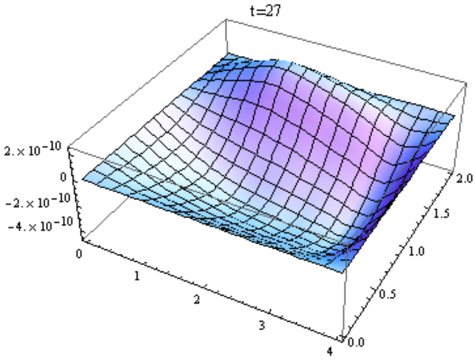

Fig. 28 Displacement for $t = 27$

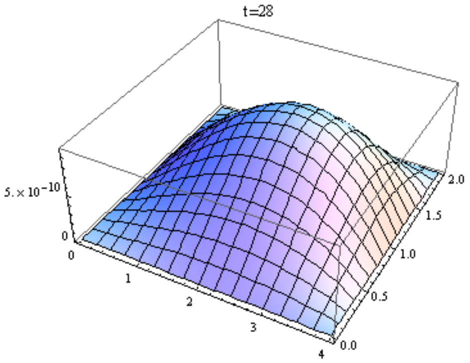

Fig. 29 Displacement for $t = 28$

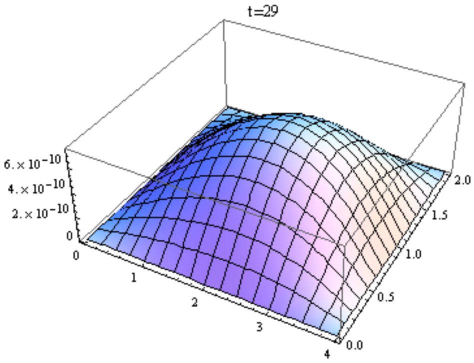

Fig. 30 Displacement for $t = 29$

References

[1] R. Portugal, I. Golebiowski and D. Frenkel, Oscillation of membranes using computer algebra, Am. J. Phys., **67** (6): 534-537 (1999).

[2] P. A. Laura, R. E. Rossi and R. H. Gutierrez, The fundamental frequency of a non-homogeneous rectangular membranes, J. Sound Vib., **204** (2): 373-37376 (1997).

[3] P. A. Laura, D. V. Bambill and R. H. Gutierrez, A note on transverse vibrations of circular, annular, composite membranes, J. Sound Vib., **205** (5): 692-697 (1997).

[4] A. Houmat, A sector Fourier p-element for free vibration analysis of sectorial membranes, Computers and Structures, **79**: 1147-1152 (2001).

[5] A. Houmat, Free vibration analysis of arbitrarily shaped membranes using the trigonometric p-version of the finite element method, Thin-Walled Structures, **44**: 943-951 (2006).

[6] Ö. Civalek, A parametric study of the free vibration analysis of rotating laminated cylindrical shells using the method of discrete singular convolution, Thin-Walled Structures, **45**: 692-698 (2007).
[7] Ö. Civalek, Free vibration and buckling analysis of composite plates with straight-sided quadrilateral domain based on DSC approach, Finite Elements in Analysis and Design, **43**: 1013-1022 (2007).
[8] E. Kreyszig, **Advanced Engineering Mathematics**, Seventh Edition, John Wiley& Sons, INC., Singapore (1993).
[9] B. D. Gupta, **Mathematical Physics,** Second Edition, Vikas Publishing House PVT LTD, New Delhi (1997).
[10] S. B. Doma, I. H. El-Sirafy, M. M. El-Borai and A. H. El-Sharif, Perturbation Treatment for the Vibrations of a Circular Membrane Subject to a Restorative Force, Alexandria Journal of Mathematics, **1** (1): 40 (2010).