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Locality of the Einstein Cosmological Constant and a Field Equation

Saad K. Nasr

Mathematics Department, Faculty of Science, Alexandria University, Alexandria, Egypt

Abstract
The Einstein cosmological constant is in fact not cosmological, and a value of it is given in terms of the curvature tensor. In the present paper we show that the curvature tensor is given in a direct form in terms of the field strength tensor, which may give a form of the equations of the field. A certain form of the field strength tensor is also introduced.

1. The Einstein Cosmological Constant
In physical cosmology, the cosmological constant (usually denoted by the Greek capital letter lambda: Λ) was proposed by Albert Einstein as a modification of his original theory of general relativity to achieve a stationary universe. Einstein abandoned the concept after the observation of the Hubble redshift indicated that the universe might not be stationary, as he had based his theory on the idea that the universe is unchanging [1]. However, the discovery of cosmic acceleration in 1998 has renewed interest in a cosmological constant.
Let

$$F^{jk} = g^{ij} \; g^{hk} \; F_{ih},$$

where (F_{ij}) is the field strength tensor and $g^{jk} \; g_{hk} = \delta^j_h$, where (g_{hk}) is a metric tensor and $i, j, h, k = 1,2,3,4,$

Consider the Einstein equation

$$R^{ij} - \frac{1}{2} \; g^{ij} \; R + \lambda \; g^{ij} = \kappa \; T^{ij}, \tag{1.1}$$

where λ is the Einstein cosmological constant, R is the scalar curvature, κ is the curvature constant, and (T^{ij}) is the energy momentum tensor, and let

$$\tau^{jk} = F^{jh} F_{ih} g^{ik} - g^{jk} F$$

where

$$4F = (F_{ih} F^{ih}).$$

Then from (1.1) and the Einstein – Maxwell's equation we can obtain (see e.g. [2], [3] Ch. 26) the equation

$$R^{jk} - 1/2\, g^{jk} R + \Lambda\, g^{jk} = \kappa\, \tau^{jk} \tag{1.2}$$

where Λ is a scalar and $j,k = 1,\ldots,4$. Let $g_{jk} = e_j\, \delta_{jk}$. Then putting $k = j$ in (1.2), multiplying the resulting equation by e_j, and adding for j from $j = 1$ to $j = 4$, we then get (loc. cit)

$$\Lambda = \frac{1}{4} R\,. \tag{1.3}$$

Hence, from (1.2) and (1.3) we get the equation

$$R^{jk} - \frac{1}{4} g^{jk} R = \kappa\, \tau^{jk}. \tag{1.4}$$

From (1.4), we get when putting $\tau^{jk} = 0$, i.e, in vacuum

$$R^{jk} = \Lambda\, g^{jk} \tag{1.5}$$

Equation (1.5) is the same as the Einstein equation in vacuum. Now, from (1.4), Einstein – Maxwell's equation can be rewritten in the form

$$R^{jk} - 1/4\, g^{jk} R = \kappa\left[(F^{jh} F_{ih} g^{ik}) - \frac{1}{4} g^{jk} (F_{ih} F^{ih}) \right] \tag{1.6}$$

We then get from (1.6) the field equation (loc. cit)

$$R^{jk} = \kappa \ (F^{jh} \ F_{ih} \ g^{ik}) \,, \tag{1.7}$$

where κ is the curvature constant, (g_{jh}) is a metric tensor and (F_{ih}) is the field strength tensor. So that (1.7) gives directly the curvature tensor (R^{jk}) in terms of the field strength tensor (F_{ih}) and the metric tensor (g_{ih}). Moreover, multiplying (1.7) by (g_{jk}) we get

$$R = 4 \, \kappa \, F \ . \tag{1.8}$$

Equation (1.8) shows that the (scalar) curvature is proportional to the (scalar) field strength.

2. The Radiant Equation

Consider the wave equation

$$\Delta \, \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4 \pi \rho \,, \qquad (\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}) \,, \tag{2.1}$$

and the radiant equation introduced by the author ([2], ch. 23)

$$\hbar \Delta \, \psi = \frac{\partial}{\partial t} (\rho \, \psi) \ , \tag{2.2}$$

where $\rho = \rho \, (\boldsymbol{x}, t) > 0$ is a given density function, $\hbar = h/2\,\pi$, and h is Planck's constant.

Now let

$$\Pi^2 = (\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})$$

and consider the 4- vector $(\boldsymbol{A}, \boldsymbol{C})$ where $\boldsymbol{A} = (A_1, A_2, A_3)$ and $c = i \, \varphi$, $(i = \sqrt{-1})$, and φ satisfies the equation (2.1), and the vector $\boldsymbol{A}$ is given by the differential equation

$$\Pi^2 \, \boldsymbol{A} \, \psi = 4 \, \pi \ \hbar / c \, \nabla \psi \,, \tag{2.3}$$

where ψ is given by the radiant equation (2.2).

Then a solution of (2.1) can be given by

$$\phi = \int \frac{[\rho]_{ret.}}{|\boldsymbol{x} - \boldsymbol{x}'|} (d^3 \boldsymbol{x}) , \tag{2.4}$$

where $[f(\boldsymbol{x},t)]_{ret.} = f(\boldsymbol{x}, t - |\boldsymbol{x}|/c), \boldsymbol{x} = (x_1, x_2, x_3).$

and

$$(d^3 \boldsymbol{x}') = dx_1' \; dx'_2$$

Also a solution of (2.3) can be given in the form

$$\mathbf{A}\psi = -\frac{\hbar}{c} \int \frac{[\nabla\psi]_{ret.}}{|\boldsymbol{x} - \boldsymbol{x}'|} \left(d^3 x'\right). \tag{2.5}$$

Now, Let $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ where ϕ and $\mathbf{A}$ are given by (2.4) and (2.5) respectively, and let

$$F_{js} = \frac{\partial \psi_s}{\partial x_j} - \frac{\partial \psi_j}{\partial x_s}, \text{ where } j, s = 1, .., 4. \tag{2.6}$$

3. Particular Metric

Let $ds^2 = f^2 \, d\, l^2 - G^2 \, d\boldsymbol{r}^2$, $(l = ct)$, where f and G are functions of $\boldsymbol{r}$ only. Then after some manipulations given in ([3] , Ch. 18) we have obtained the equations

$$4G^{\frac{3}{2}} \, \Delta(\sqrt{G}) = \kappa \, |B|^2 \tag{3.1}$$

and

$$\frac{f}{G} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(G \frac{\partial f}{\partial x_j}\right) = \kappa \, |E|^2, \tag{3.2}$$

where κ is the curvature constant, and

$$|E|^2 = E_1^2 + E_2^2 + E_3^2, \; |B|^2 = B_1^2 + B_2^2 + B_3^2,$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\mathrm{a}$$

and

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

Equation (3.1) gives G in terms of the square of the length of the (magnetic) vector **B**, and then (3.2) gives the function f in terms of the square of the length of the (electric) vector **E**. In view of the smallness of the curvature constant κ the changes in the functions G and f from their Euclidean value one for each of them, may be represented by the equations

$$\Delta(\sqrt{G}) = (\kappa/4)\,|B|^2 \tag{3.3}$$

and

$$div(G\nabla f) = \kappa|\mathrm{E}|^2 \tag{3.4}$$

However, a particular form of the density function $\rho(x,t)$ is considered, namely, ρ has been taken to be the gravitational density $\frac{\mu}{r^2}$ where μ is the Newtonian gravitational constant ([3] or [4] Ch. 18), $r \le \mu/\hbar$. Then taking into consideration the smallness of the constants κ and ħ the function G has been obtained in the form (loc.cit.)

$$\sqrt{G} = 1 + \frac{M}{r},\ (M \text{ is a constant} > 0). \tag{3.5}$$

And the function f has been also obtained from (3.4) and (3.5) in the form

$$f = 1 + \kappa\,(4\pi\,\mu)^2\left\{\log(r+M) + \frac{M}{r+M}\right\}. \tag{3.6}$$

So that, if M is small relative to r we may take in view of (3.6), the function f to be of the following form

$$f = 1 + \kappa\,(4\pi\mu)^2\log\,(r+M). \tag{3.7}$$

Equation (3.7) is essentially different from the corresponding value of f given in the metric of Schwarzchild.

References

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