Some Dynamic Properties of a Discontinuous Dynamical System

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Abstraqct
In this work we are concerned with the definition and some properties of the discontinuous dynamical systems. Then, we study a discontinuous dynamical system. The existence, uniqueness and the continuous dependence on the initial data of the solution will be proved. The local stability at the equilibrium points will be studied. The bifurcation analysis and chaos will be discussed.

Keywords: Functional equations, discontinuous dynamical systems, existence, uniqueness, continuous dependence, equilibrium points, local stability, bifurcation, chaos.

1 Introduction
Let R^+ be the set of positive real numbers and let $r \in R^+$. Consider the problem of retarded functional equation

$$x(t) = f(t, x(t-r)), \quad t \in (0, T], \tag{1.1}$$

$$x(t) = x_0, \quad t \le 0. \tag{1.2}$$

Now, if T be positive integer, $r = 1$, and $t = n = 1,2,3,\cdots,T$ is a constant, then the problem (1.1)-(1.2) will be the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1,2,3,\cdots,T \tag{1.3}$$

$$x_n = x_0, \quad t \le 0. \tag{1.4}$$

This shows that the discrete dynamical system (1.3)-(1.4) is a special case of the problem of retarded functional equation (1.1)-(1.2).

2 Discontinuous Dynamical Systems

Consider the problem

$$x(t) = f(x(t-r)), \quad t \in (0,T] \tag{2.1}$$

$$x(t) = x_0, \quad t \le 0.$$

Let $t \in (0,r]$, then $t-r \in (-r,0]$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_r(t) = f(x_0), \ t \in (0,r].$$

For $t \in (r,2r]$, we find that $t-r \in (0,r]$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_{2r}(t) = f(x_r(t)) = f(f(x_0)) = f^2(x_0), \quad t \in (r,2r].$$

Repeating the process we can deduce that the solution of the problem (1.1)-(1.2) is given by

$$x(t) = x_{nr}(t) = f^n(x_0), \quad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr)$, $k = 1,2,..., n$, but

$$\lim_{t \to kr^+} x_{(k+1)r}(t) = f^{k+1}(x_0) \ne x_{kr}(t),$$

which implies that the solution of the problem (1.1)-(1.2) is discontinuous (sectionally continuous) on $(0,T]$ and we have proved the following theorem,

Theorem 2.1
The solution of the problem of retarded functional equation (1.1)-(1.2) is discontinuous (sectionally continuous) even the function f is continuous.

Now, let $f:[0,T] \times R^n \to R^n$ and $r_1, r_2, ..., r_n \in R^+$. Then we can give the following definitions,

Definition 2.1
The discontinuous dynamical system is the problem of retarded functional equation

$$x(t) = f(t, x(t-r_1), x(t-r_2), ..., x(t-r_n)), \quad t \in (0,T], \tag{2.2}$$

$$x(t) = x_0, \ \ t \le 0 \tag{2.3}$$

Definition 2.2
The equilibrium points of the discontinuous dynamical system (2.2)-(2.3) is the solutions of the equation,

$$x(t) = f(t, x, x, ..., x).$$

Consider now the discontinuous dynamical system of the Riccati type equation

$$x(t) = 1 - \rho x^2(t-r), \quad t \in (0,T], \tag{2.4}$$

$$x(t) = x_0, \quad t \le 0. \tag{2.5}$$

We study here the existence of a unique continuously dependent solution of the problem (2.4)-(2.5). The asymptotic stability (see [1]- [9]) at the equilibrium points will be studied. To study bifurcation and chaos, we take firstly $r = 1$ and we compare the results with the results of the discrete dynamical system of Ricatti type difference equations,

$$x_n = 1 - \rho\, x_{n-1}^2, \quad n = 1,2,.... \tag{2.6}$$

Secondly, we take some other values of r and T and study some examples.

3 Existence and Uniqueness

Let $L^1 = L^1[0,T],\ T < \infty$ be the class of Lebesgue integrable functions on $[0,T]$ with norm

$$\| f \| = \int_0^T | f(t) | dt, \ f \in L^1. \tag{3.1}$$

Let $D = \{ x \in R : 0 \le x(t) \le 1,\ t \in (0,T]\ and\ x(t) = x_0,\ t \le 0 \}$.

Definition 3.1

By a solution of the problem (2.4)-(2.5) we mean a function $x \in L^1$ satisfying the problem (2.4)-(2.5).

Now we have the following theorem,

Theorem 3.1

The problem (2.4)-(2.5) has a unique solution $x \in L^1$.

Proof

Define, on D, the operator $F : L^1 \to L^1$ by

$$Fx(t) = 1 - \rho\, x^2(t-r). \tag{3.2}$$

The operator F makes sense, indeed for $x \in D$ we have

$$| Fx(t) | \le 1 + \rho\, | x(t-r) | \tag{3.3}$$

and

$$\| Fx \| = \int_0^T | Fx(t) | \, dt \le \int_0^T (1 + \rho \, | x(t-r) |) dt$$

$$\le T + | \rho | \left(\int_0^r x_o \, dt + \int_r^T | x(t-r) | \, dt \right)$$
$$\le T + r \, (x_0 + \| x \|). \tag{3.4}$$

Now for $x, y \in D,$ we can obtain,

$$| Fx - Fy | = | \rho x^2 (t-r) - \rho y^2 (t-r) |$$

$$\le 2 \, \rho \, | x(t-r) - y(t-r) |$$

which implies that,

$$\| Fx - Fy \| \le 2 \, \rho \int_0^T | x(t-r) - y(t-r) | \, dt$$

$$= 2 \, \rho \left[\int_0^r | x(t-r) - y(t-r) | \, dt + \int_r^T | x(t-r) - y(t-r) | \, dt \right] \tag{3.5}$$

$$= 2 \, \rho \left[\int_r^T | x(t-r) - y(t-r) | \, dt \right] \le 2 \, \rho \, | x - y | .$$

If $2 \, \rho < 1,$ we deduce that,

$$\| Fx - Fy \| \; < \; \| x - y \| \tag{3.6}$$

and then the problem (1.1)-(1.2) has, on $D,$ a unique solution $x \in L^1.$

4 Continuous Dependence on Initial Conditions

Now we study the continuous dependence of the solution of the problem (1.1)-(1.2) on the initial data x_0 .

Theorem 4. 1

If $2 \, \rho < 1.$ Then the solution of the discontinuous dynamical system (1.1)-(1.2) is continuously dependent on the initial data in the sense that,

$$| x_0 - x_0^* | \le \delta \;\; \Rightarrow \;\; \| x - x^* \| \le \varepsilon$$

where x^* is the solution of the problem

$$x(t) = \rho \, x^2 (t-r), \quad t \in (0, T],$$

$$x(t) = x_0^*, \quad t \le 0. \tag{4.1}$$

Proof

Let x and x^* be the two solutions of the two problems (1.1)-(1.2) and (1.1)-(4.1) respectively, then

$$|x(t)-x^*(t)|\le \rho |x^2(t-r)-x^{*^2}(t-r)|$$

which implies that

$$|x(t)-x^*(t)|\le 2\rho |x(t-r)-x^*(t-r)|,$$

then

$$\| x-x^* \| \le 2\rho |x_o - x_o^*| + 2\rho \| x-x^* \|$$

and

$$\| x-x^* \| \le \frac{2\rho}{1-2\rho} |x_o - x_o^*| .$$

Hence

$$|x_0 - x_0^*| \le \delta \ \Rightarrow \ \| x-x^* \| \le \delta \frac{2\rho}{1-2\rho} = \varepsilon$$

and the result follows.

5 Equilibrium Points and Asymptotic Stability

The equilibrium points of (2.4) are the solution of the equation

$$1-\rho x_{eq}^2 = x_{eq}$$

which are

$$(x_{eq})_1 = \frac{-1-\sqrt{1+4\rho}}{2\rho},$$

$$(x_{eq})_2 = \frac{-1+\sqrt{1+4\rho}}{2\rho}.$$

The equilibrium point of (2.4) is locally asymptotically sable if all the roots λ of the equation

$$\lambda^r + 2\rho x_{eq} = 0, \qquad (5.1)$$

satisfy $|\lambda| < 1$ (see [10]).

Then, the equilibrium point $x_{eq} = \frac{-1-\sqrt{1+4\rho}}{2\rho}$ is locally asymptotically stable if all the roots λ of the equation

$$\lambda^r - (1+\sqrt{1+4\rho}) = 0. \qquad (5.2)$$

satisfy $|\lambda| < 1$.

While the second equilibrium point $x_{eq} = \frac{-1+\sqrt{1+4\rho}}{2\rho}$ is locally asymptotically stable if all the roots λ of the equation

$$\lambda^r - (1 - \sqrt{1 + 4\rho}) = 0. \quad (5.3)$$

satisfy $|\lambda| < 1$.

In studying (2.4) it may be useful to study the difference equation (2.6).

6 Bifurcation and Chaos

In this section, some numerical simulation results are presented to show that dynamics behaviors of the discontinuous dynamical system (2.4)-(2.5) change for different values of r and T. To do this, we will use the bifurcation diagrams as follow:-

Example:

1. we take $r = 1$ and $t \in [0,30]$, in this case, we get the same behavior as in the discrete case (2.6) (Figure 1).
2. we take $r = 2$ and $t \in [0,30]$ (Figure 2).
3. we take $r = 1.5$ and $t \in [0,30]$ (Figure 3).
4. we take $r = 0.1$ and $t \in [0,3]$ (Figure 4).
5. we take $r = 0.2$ and $t \in [0,3]$ (Figure 5).
6. we take $r = 0.3$ and $t \in [0,3]$ (Figure 6).

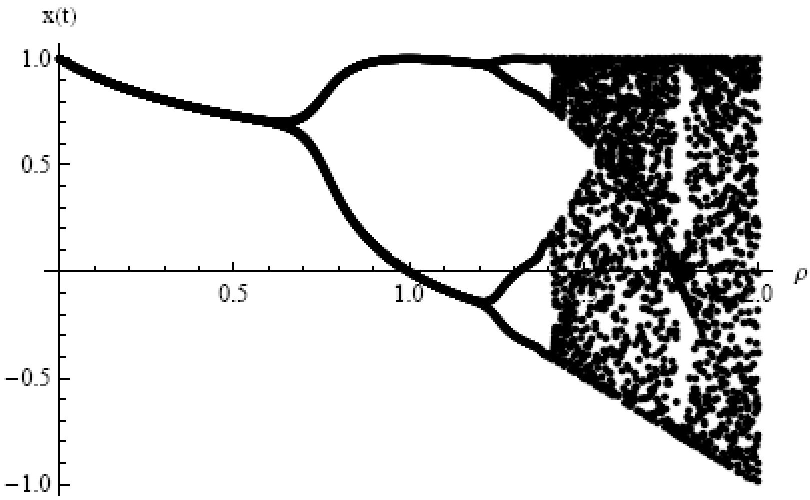

Figure 1: Bifurcation diagram of map (2.4)-(2.5) with respect to ρ, $r = 1$ and $t \in [0,30]$.

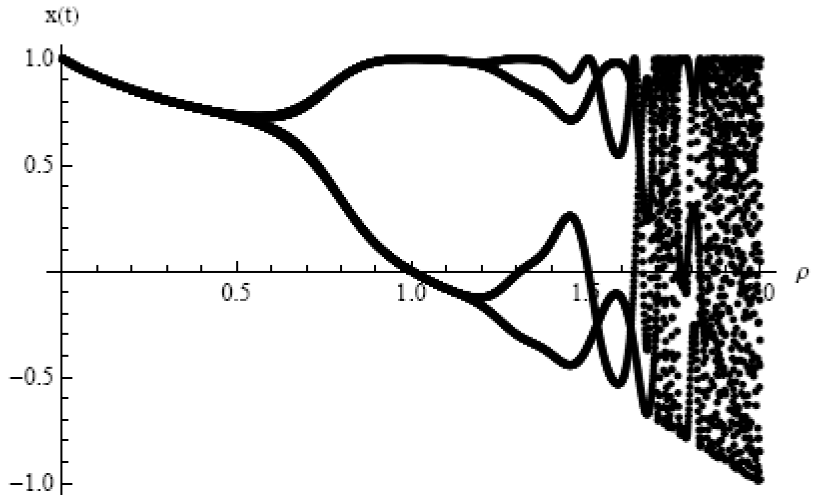

Figure 2: Bifurcation diagram of map (2.4)-(2.5) with respect to ρ, $r = 2$ and $t \in [0,30]$.

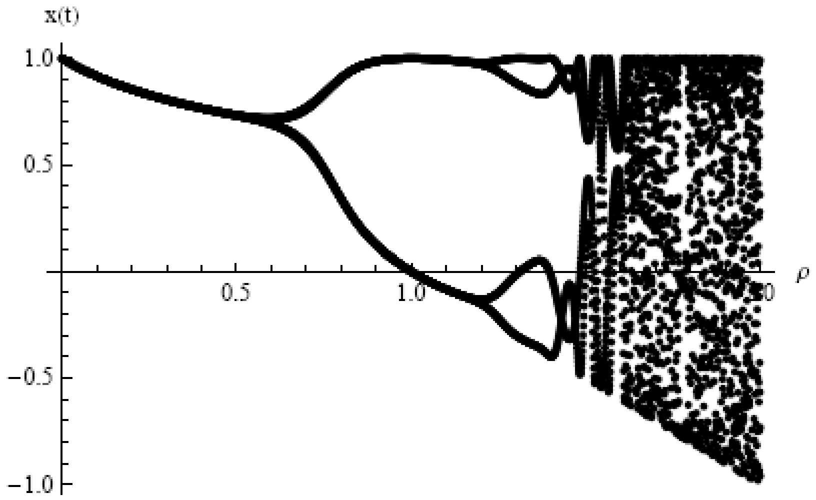

Figure 3: Bifurcation diagram of map (2.4)-(2.5) with respect to ρ, $r = 1.5$ and $t \in [0,30]$.

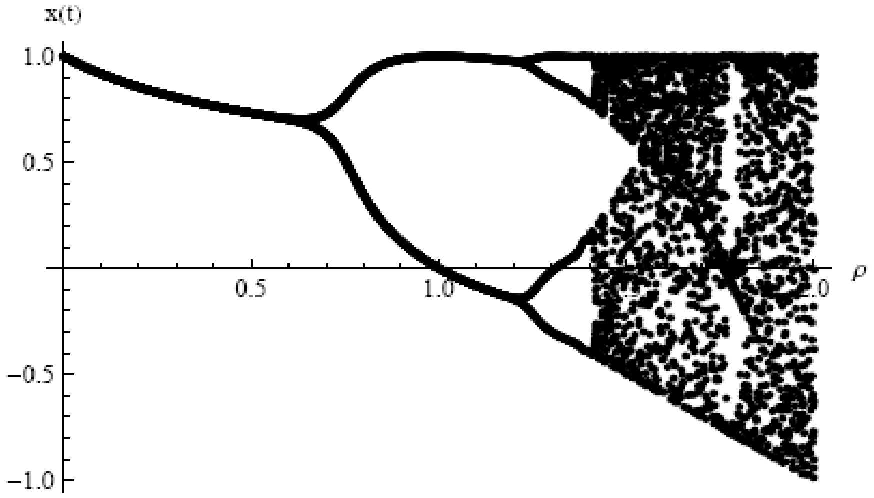

Figure 4: Bifurcation diagram of map (2.4)-(2.5) with respect to ρ, $r = 0.1$ and $t \in [0,3]$.

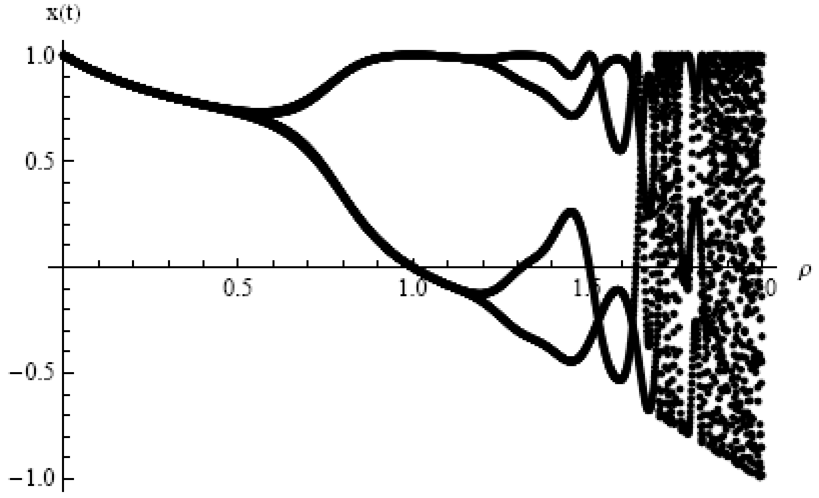

Figure 5: Bifurcation diagram of map (2.4)-(2.5) with respect to ρ, $r = 0.2$ and $t \in [0,3]$.

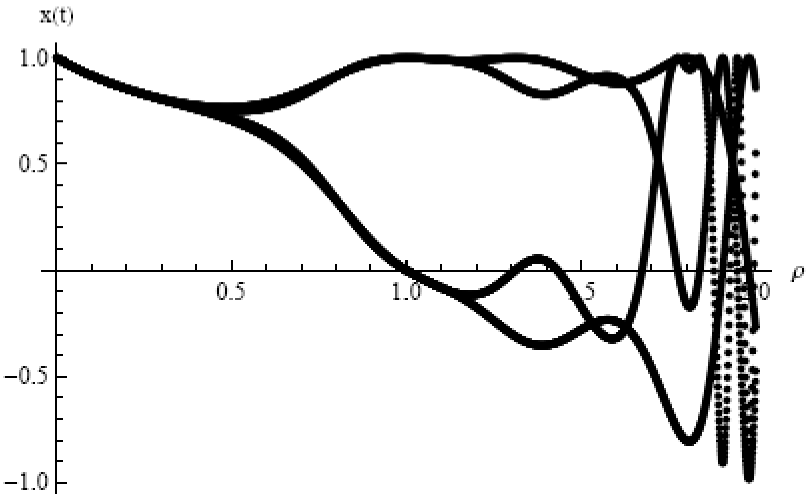

Figure 6: Bifurcation diagram of map (2.4)-(2.5) with respect to ρ, $r = 0.3$ and $t \in [0,3]$.

From Figures (1 - 6) we deduce that the change of r and T effect of stability of the Logistic equation model, occurs of a bifurcation point, parameter sets for which aperiodic behavior occur and parameter sets for which a chaotic behavior occur.

7 Conclusions

The discrete dynamical system model describes the dynamical properties for the case $r = 1$ and the time is discrete $t = 1,2,3,4,\cdots$.

On the other hand, the discontinuous dynamical system model describes the dynamical properties for different values of the delayed parameter $r \in R^+$ and the time $t \in [0,T]$ is continuous.

Figures (1), (2) agree with the results of the asymptotic stability, this confirm that our numerics are correct. Also from figures (1), (4) and (2), (5), it locks like that there is a scale that gives identical chaos behavior.

This shows the richness of the models of discontinuous dynamical systems.

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