Fractional Parallel RLC Circuit

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Abstract
In this paper, the methodology of fractional calculus is applied to the electric parallel RLC circuit. The existence and uniqueness of the solution of the model are proved. The mathematical model is solved by both Adomian decomposition and Laplace transform methods. A brief comparison between these methods is introduced.

Keywords: Fractional calculus; Electric circuits; Adomian decomposition method; Laplace transform method.

1. Introduction
Fractional derivatives provide excellent instruments for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other applications in the fields of physics, signal processing, fluid mechanics, mathematical biology, and bioengineering [1-5]. The other large field which requires the use of derivatives of non-integer order is the recently elaborated theory of fractals, has opened farther perspectives for the theory of fractional derivatives, especially in modeling dynamical processes in self-similar and porous structures [6–9]

Capacitors and inductors are the main elements in analogue circuits and are used comprehensively in many electronic systems. However, the ideal capacitor and ideal inductor cannot exist in nature, because the impedance of the capacitor of form $1/(j\omega C)$ and the impedance of the inductor of the form $(j\omega L)$ would violate the causality. In fact, the dielectric materials exhibit a fractional behavior yielding the impedance of the real capacitor of the form $1/[(j\omega C)^{\alpha}]$ and the impedance of the real inductor of the form$[(j\,\omega L)^{\beta}]$, with $\alpha, \beta \in \Re^{+}$ [10-14].

2. Fractional Calculus Definitions and Concepts

This section introduces some basic definitions and properties of fractional calculus theory [1], [10] and [11]

Let $L^1 = L^1[a,b]$, $0 \le a,\ b < \infty$ be class of Lebesgue integralable functions on $[a,b]$.

Definition 1

The Riemann-Liouville integral operator of order $\alpha \ge 0$ of the function $v(t) \in L^1$ is defined by

$$J_a^{\alpha} v(t) = \int_a^t \frac{(t-s)^{\alpha-1} v(s)}{\Gamma(\alpha)} ds \tag{2.1}$$

where $\Gamma(\alpha)$ is the gamma function. As a special case, when $a=0$, we write $J_0^{\alpha} = J^{\alpha}$

Definition 2

The Riemann-Liouville fractional derivative of $v(t)$ of order α $m-1<\alpha<m$, is defined by

$$^{R}D_a^{\alpha} v(t) = \frac{d^m}{dt^m} J_a^{m-\alpha} v(t) \tag{2.2}$$

Definition 3

The Caputo fractional derivative of order $\alpha \in (m-1, m]$, $m = 1, 2, 3, \ldots$, of the absolutely continuous function $v(t)$ is defined by

$$D_a^{\alpha} v(t) = J_a^{m-\alpha} \frac{d^m}{dt^m} v(t) = \int_a^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} v^{(m)}(s)\, ds \tag{2.3}$$

When $a=0$ we can write $D_0^{\alpha} = D^{\alpha}$.

Definition 4

The Laplace transform of the function $v(t)$, denoted by $V(s)$, is defined by the equation

$$V(s) = \mathrm{L}[v(t)] = \int_0^{\infty} e^{-st} v(t)\, dt \tag{2.4}$$

Laplace transform of Caputo derivative is

$$\mathrm{L}\left\{D^{\alpha} v(t)\right\} = s^{\alpha} V(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} v^{(k)}(0)\Big|_{t=0}, \quad n = \lceil \alpha \rceil \tag{2.5-a}$$

Laplace transform of the Riemann-Liouville integral

$$\mathrm{L}\left\{J^{\alpha} v(t)\right\} = s^{-\alpha} V(s) \tag{2.5-b}$$

3. Fractional Order Parallel RLC Circuit

Our aim here is to generalize the model of parallel RLC circuit by using the following Caputo derivative: (for more details see [10], [12] and [13])

$$i_c(t) = C\ D^{\beta} v_c(t),\ 0<\beta<1,\ t>0 \tag{3.1}$$

$$v_l(t) = L\ D^{\alpha} i(t),\ 0<\alpha<1,\ t>0 \tag{3.2}$$

Consider the circuit of figure (1) consisting of resistor R, capacitor C and inductor L which are connected in parallel, where $i_l(0)=I_0$, is the initial inductor current, $v_c(0)=v_0$ is initial capacitor voltage, $u_0(t)$ is the unit step function and $i_s(t)$ is the current source [19].

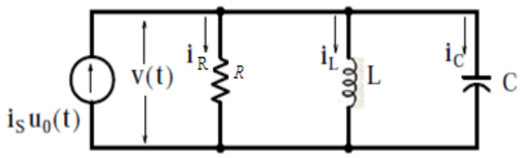


Figure (1)

This circuit can be modeled by the following fractional order integro-differential equation

$$C\ D_t^{\beta} v(t)+\frac{1}{L}J^{\alpha}v(t)+\frac{1}{R}v(t)+I_0=i_s(t),\ t>0 \tag{3.3}$$

By using equation (3.1), we can transform equation (3.3) to the following fractional order integral equation

$$i_c(t)=\frac{-1}{LC}J^{\alpha+\beta}\ i_c(t)-\frac{1}{RC}J^{\beta}\ i_c(t)-(\frac{1}{L}J^{\alpha}v_c(0)+\frac{1}{R}v_c(0)+I_0-i_s(t)) \tag{3.4}$$

We write

$$i(t)=f\ (t\ ,\ J^{\beta}i(t)\ ,\ J^{\alpha+\beta}i(t))=-k_2 J^{\alpha+\beta}\ i(t)-k_1\ J^{\beta}\ i(t)-(\frac{1}{L}J^{\alpha}v_c(0)+\frac{1}{R}v_c(0)+I_0-i_s(t))$$

where $i(t)=i_c(t),\ k_1=\dfrac{1}{RC},\ k_2=\dfrac{1}{LC},\ \alpha\in(0,1],\ \beta\in(0,1]$

3.1 The state equation of the circuit

We can describe parallel RLC circuit by the state equations of the form [15]

$$\begin{bmatrix} \dfrac{d^{\alpha}x_1(t)}{dt^{\alpha}} \\ \dfrac{d^{\beta}x_2(t)}{dt^{\beta}} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & \dfrac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{C} \end{bmatrix} \left[i_s(t) \right] \tag{3.5}$$

where $x_1(t)=i_l(t)$ and $x_2(t)=v_c(t)$.

Assuming that the fractional order for inductor and capacitor are equal, i.e. $\alpha=\beta=q$ the state equation may be described by the following equation:

$$\frac{d^q x(t)}{dt^q} = A\,x(t) + B\,i_s(t), \quad 0 < q < 1. \tag{3.6}$$

The solution of equation (3.6) may be obtained as follows:

$$x(t) = \Phi_0(t)x_0 + \int_0^{\infty} \Phi(t-s)\,B\;i_s(s)\,ds, \quad x(0) = x_0 \tag{3.7}$$

where

$$\Phi_0(t) = E_q(At^q) = \sum_{m=0}^{\infty} \frac{A^m t^{mq}}{\Gamma(mq+1)}, \tag{3.8}$$

$$\Phi(t) = \sum_{m=0}^{\infty} \frac{A^m t^{(m+1)q-1}}{\Gamma((m+1)q)} \tag{3.9}$$

and

$$i_c(t) = i_s(t) - i_R(t) - i_l(t) \tag{3.10}$$

where $E_q(At^q)$ denotes the Mittage-Leffler matrix function.

3.2 Existence and uniqueness of the solution

Define the mapping $F: E \to E$ where E is the Banach space $(C[0,T], \|.\|)$, the space of all continuous functions on $[0,T]$ with the norm

$\| y \| = sup_{t\in[0,T]}|y(t)|$

Theorem

Let f satisfies the Lipschitz condition [2] then problem (3.4) has a unique solution $i \in C[0,T]$

Proof

The mapping $F: E \to E$ is defined as

$$Fi = f\,(t,\; J^{\beta}i,\; J^{\alpha+\beta}i), \qquad i, y, z \in C[0,T]$$

then, $Fy - Fz = f\,(t, J^{\beta}y, J^{\alpha+\beta}y) - f\,(t, J^{\beta}z, J^{\alpha+\beta}z)$

This implies that

$$|Fy - Fz| \leq k_1 |J^{\beta}y - J^{\beta}z| + k_2 |J^{\alpha+\beta}y - J^{\alpha+\beta}z|$$

$$|Fy - Fz| \leq \frac{k_1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |y-z|\,ds + \frac{k_2}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |y-z|\,ds$$

$$|Fy - Fz| \le \frac{k_1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sup_{t\in[0,T]} |y-z|\, ds + \frac{k_2}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \sup_{t\in[0,T]} |y-z|\, ds$$

$$|Fy - Fz| \le \frac{k_1}{\Gamma(\beta)} \| y-z \| \int_0^t (t-s)^{\beta-1}\, ds + \frac{k_2}{\Gamma(\alpha+\beta)} \| y-z \| \int_0^t (t-s)^{\alpha+\beta-1} ds$$

$$|Fy - Fz| \le \| y-z \| \left(\frac{k_1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\, ds + \frac{k_2}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} ds \right)$$

$$|Fy - Fz| \le \| y-z \| \left(\frac{k_1 T^{\beta}}{\Gamma(\beta+1)} + \frac{k_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right)$$

If $k = \left(\frac{k_1 T^{\beta}}{\Gamma(\beta+1)} + \frac{k_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) < 1$ we get

$\| Fy - Fz \| < k \| y - z \|$

Therefore, the mapping F is a contraction and there exists a unique solution $i \in C[0,T]$ to the equation (3.4).

4. Approximate Solution of the Model Using Different Methods

In this section the Laplace transform and Adomian decomposition methods will be used to solve the model.

4.1 Laplace transform method (LTM)

Applying LTM to both sides of equations (3.3) and (3.4), we get

$$V_c(s) = \frac{s^{\alpha+\beta-1}\, v_c(0) + \frac{1}{C} s^{\alpha} I_s(s) - \frac{1}{C} I_0\, s^{\alpha-1}}{s^{\alpha+\beta} + \frac{1}{RC} s^{\alpha} + \frac{1}{LC}} \qquad (4.1)$$

$$I_c(s) = \frac{C\, v_c(0)\, s^{\alpha+2\beta-1} + s^{\alpha+\beta} I_s(s) - I_0\, s^{\alpha+\beta-1}}{s^{\alpha+\beta} + \frac{1}{RC} s^{\alpha} + \frac{1}{LC}} - C\, v_c(0)\, s^{\beta-1} \qquad (4.2)$$

We then, use FORTRAN code to evaluate the inverse Laplace transform of (4.1) and (4.2) to get $v_c(t)$ and $i_c(t)$ at different values of α and β numerically.

4.2 Adomian Decomposition Method (ADM)

By using ADM, the solution is obtained as an infinite series in which each term can be easily obtained using the preceding terms that converge rapidly towards the accurate solution. According to ADM, we can deduce from (3.4) the following recurrence relation [17, 18]:

$$i_0(t) = -\frac{1}{L} J^{\alpha} v_c(0) - \frac{1}{R} v_c(0) - I_0 + i_s(t) \tag{4.3}$$

$$i_{n+1}(t) = -\frac{1}{LC} J^{\alpha+\beta} i_c(t) - \frac{1}{RC} J^{\beta} i_c(t) \tag{4.4}$$

And the solution of equation (3.4) will be

$$i_c(t) = \sum_{j=0}^{\infty} i_j(t), \; v_c(t) = \frac{1}{C} J^{\beta} i_c(t) + v_c(0) \tag{4.5}$$

5. Numerical Examples

In this section we will introduce three examples to describe the under damped, over damped and critical damped response.

Example (1): Circuit under damped response

In this example, we will use the following data [19]: $i_l(0)=0$, $v_c(0)=0$, $R=100\,\Omega$, $L=10\,H$, $C=1500\,\mu F$, $i_s(t)=10\,u_0(t)$

The following figures represent the capacitor current and the capacitor voltage at different values of α and β

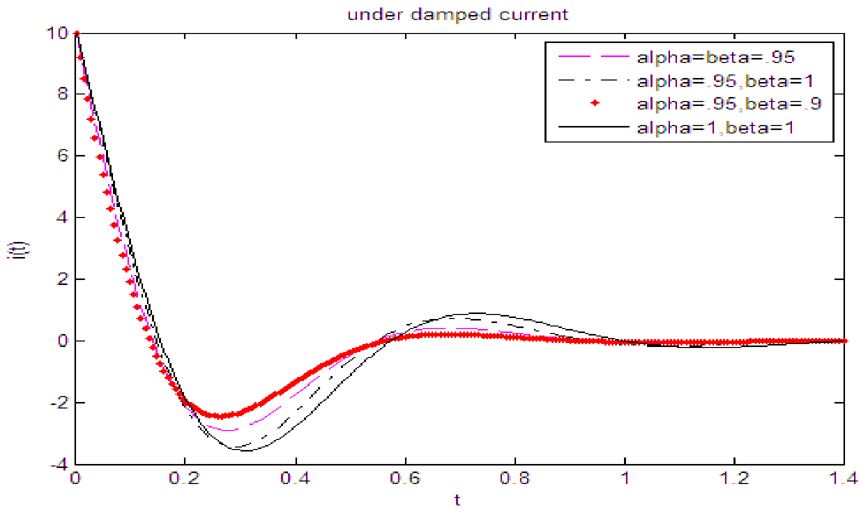


Figure (2) the capacitor current $i_c(t)$ by LTM

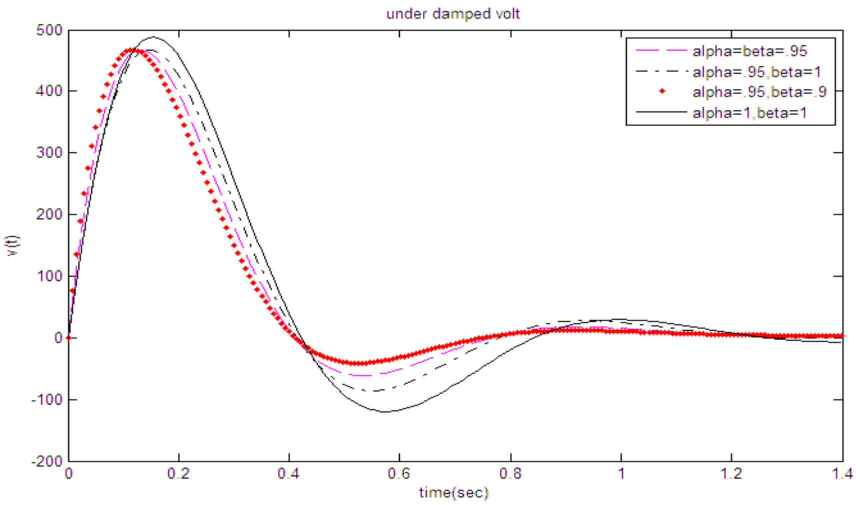


Figure (3) the capacitor voltage $v_c(t)$ by LTM

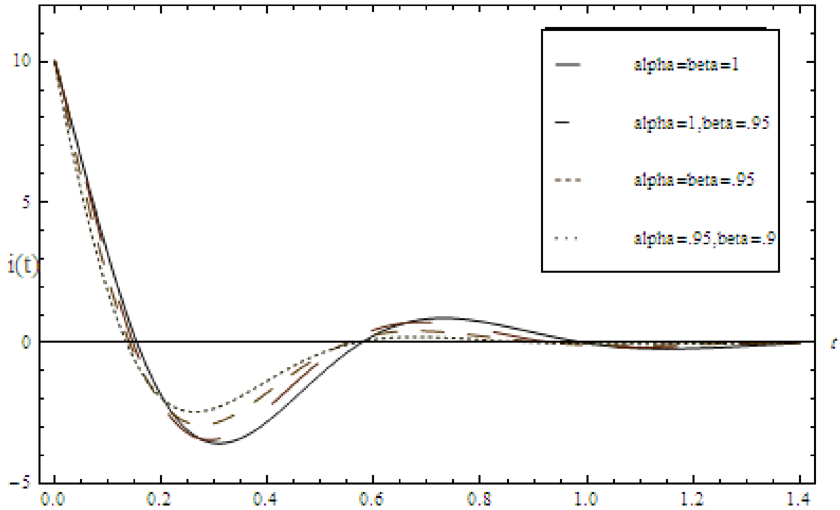


Figure (4) the capacitor current $i_c(t)$ by ADM

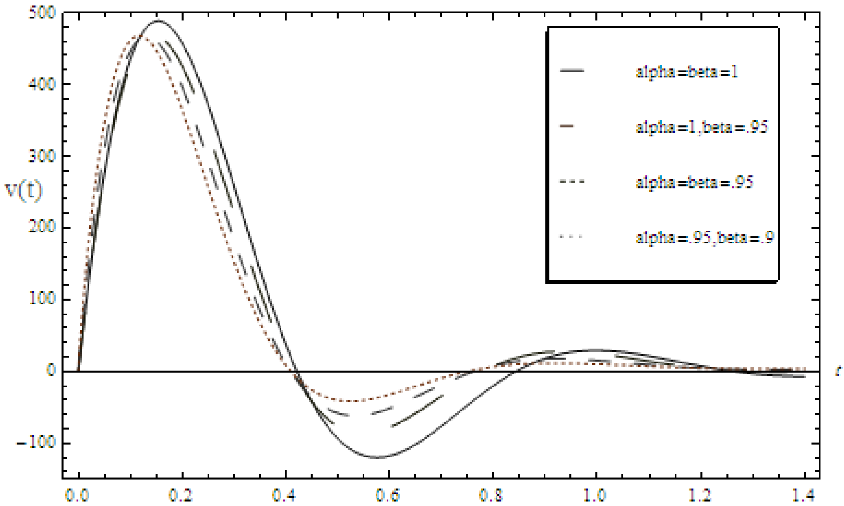


Figure (5) the capacitor voltage $v_c(t)$ by ADM

Tables (1) and (2) show a comparison between the absolute error of LTM solution and ADM solution at $\alpha = \beta = 0.95$

Table (1) Absolute error of the capacitor current $i_c(t)$

t	Error of LTM	Error of ADM
.00001	3.00324×10^{-7}	3.24022×10^{-10}
.09801	0.0000194378	3.78222×10^{-8}
.20301	0.0000100606	3.94229×10^{-8}
.40601	1.3471×10^{-6}	4.7104×10^{-8}
.60201	1.21259×10^{-6}	1.25948×10^{-8}
0.8050	5.13375×10^{-7}	1.33746×10^{-8}
1.0010	1.40035×10^{-7}	4.00347×10^{-8}
1.2040	1.96007×10^{-7}	4.03993×10^{-7}
1.4000	4.96846×10^{-8}	7.74968×10^{-6}

Table (2) Absolute error of the capacitor voltage $v_c(t)$

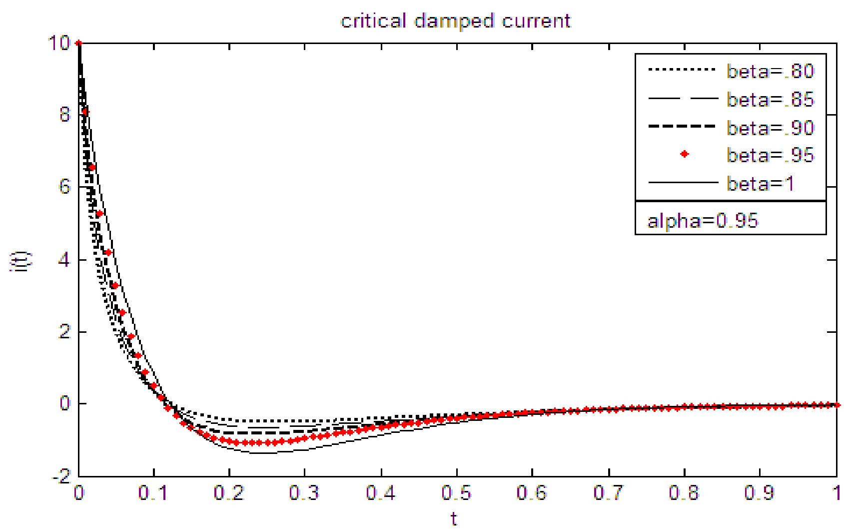
t	Error of LTM	Error of ADM
.00001	1.30124×10^{-8}	4.61244×10^{-9}
.09801	0.00041708	1.97342×10^{-8}
.20301	0.000823928	2.76396×10^{-8}
.40601	0.000100992	8.07493×10^{-9}
.60201	0.0000776872	1.27829×10^{-8}
0.8050	0.0000385886	8.86489×10^{-8}
1.0010	6.69449×10^{-6}	5.94488×10^{-7}
1.2040	0.0000122533	0.0000150533
1.4000	1.81089×10^{-7}	0.000414719

Example (2): Circuit under critical damped response

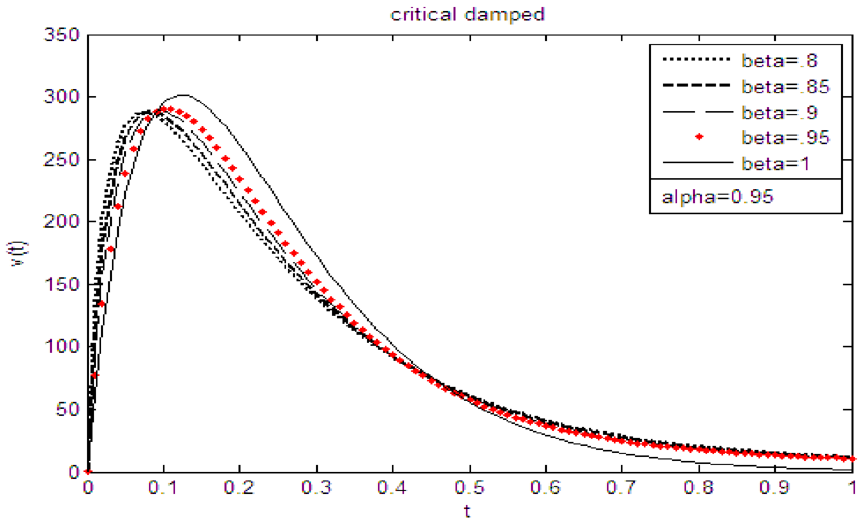
In this example, we will use the following data [19]:

$R = 41\,\Omega,\ L = 10H,\ C = 1500\mu F,\ i_l(0) = 0$, and $v_c(0) = 0,\ i_s(t) = 10\,u_0(t)$,

The following figures represent the capacitor current and the capacitor voltage at different values of α and β



Figures (6) the capacitor current $i_c(t)$ by LTM



Figures (7) the capacitor voltage $v_c(t)$ by LTM

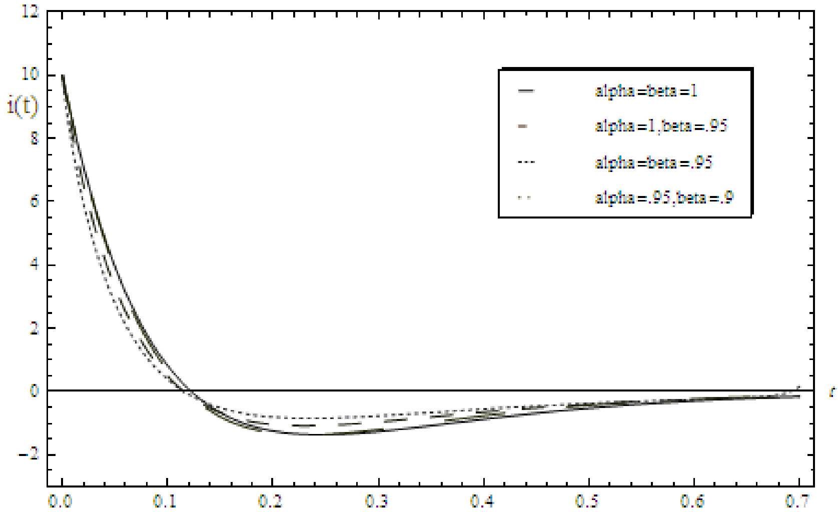


Figure (8) the capacitor current $i_c(t)$ by ADM

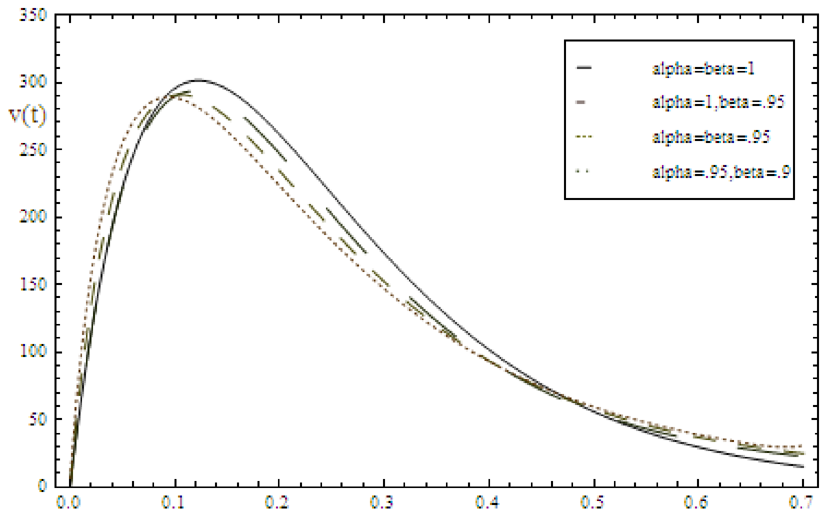


Figure (9) the capacitor voltage $v_c(t)$ by ADM

Tables (3) and (4) show a comparison between the absolute error of LTM solution and ADM solution at $\alpha = \beta = 0.95$

Table (3) Absolute error of the capacitor current $i_c(t)$

t	Error of LTM	Error of ADM
.06	0.0000421898	0.0000703898
.09	0.0000380383	0.0000422383
.15	0.0000198239	0.0000265239
.20	6.18606×10^{-6}	6.18606×10^{-6}
.30	8.70271×10^{-6}	8.60271×10^{-6}
.40	0.0000116232	0.0000116232
.50	9.91911×10^{-6}	9.91911×10^{-6}
.60	4.78444×10^{-6}	4.88444×10^{-6}
.70	2.16341×10^{-6}	8.76341×10^{-6}
.80	8.17929×10^{-7}	0.00602672

Table (4) Absolute error of the capacitor voltage $v_c(t)$

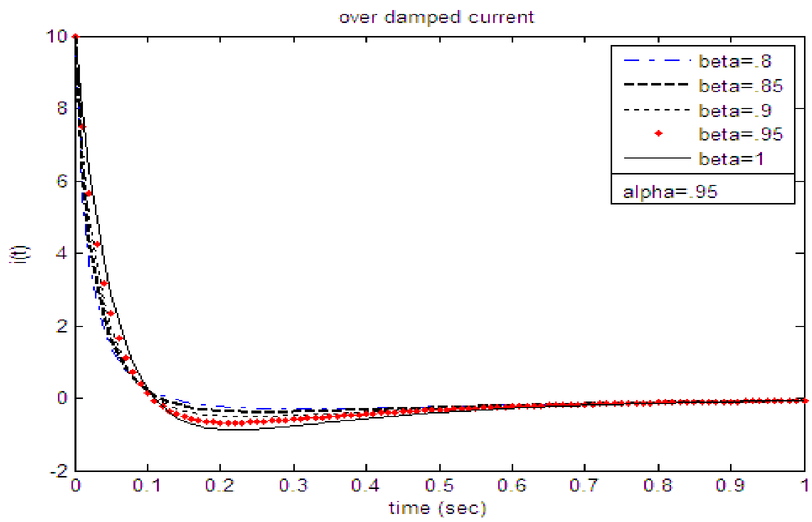
t	Error of LTM	Error of ADM
.06	0.000973406	0.00166401
.09	0.000344232	0.000410432
.15	0.00100491	0.00131311
.20	0.00175884	0.00173184
.30	0.00216433	0.00215543
.40	0.00182901	0.00183661
.50	0.00134553	0.00135523
.60	0.000636307	0.000623407
.70	0.000281337	0.000185037
.80	0.000115598	0.075318

Example (3): Circuit under over damped response

In this example, we will use the following data [19]:

$i_l(0) = 0$, and, $v_c(0) = 0$, $R = 30\ \Omega$, $L = 10H$, $C = 1500\ \mu F$, $i_s(t) = 10\, u_0(t)$

The following figures represent the capacitor current and the capacitor voltage at different values of α and β



Figures (10) the capacitor current $i_c(t)$ by LTM

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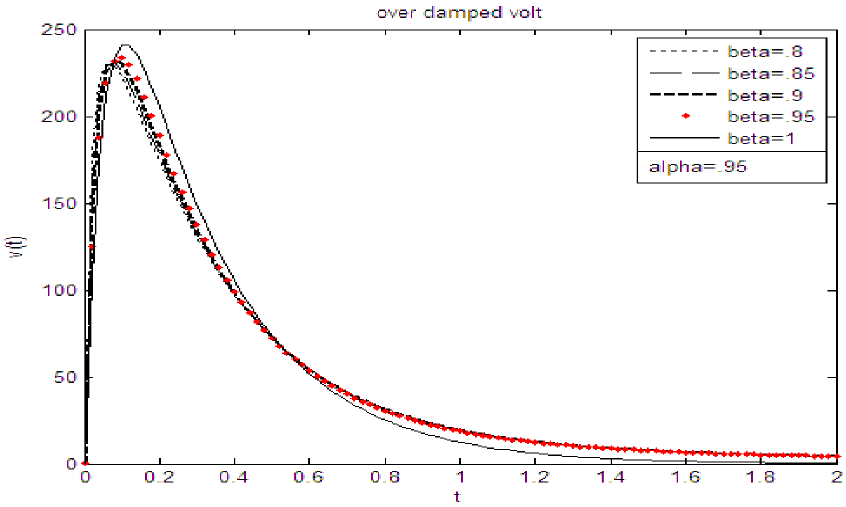

Figures (11) the capacitor voltage $v_c(t)$ by LTM

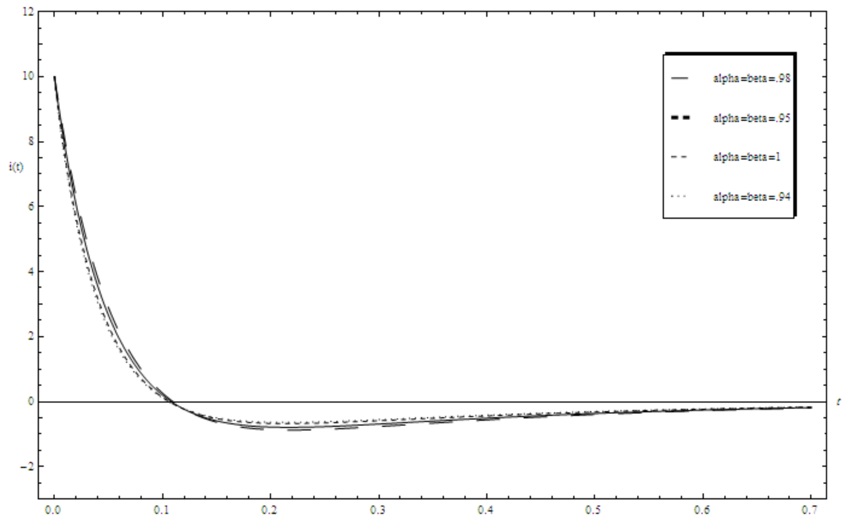

Figure (12) the capacitor current $i_c(t)$ by ADM

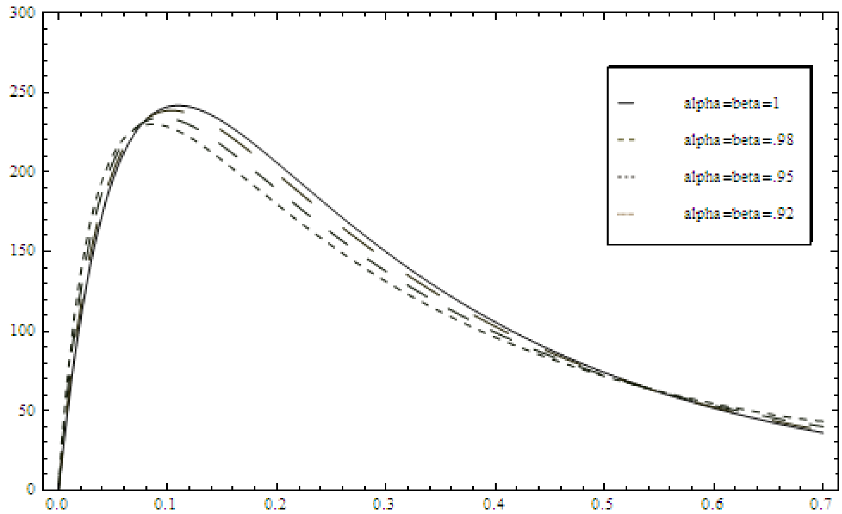

Figure (13) the capacitor voltage $v_c(t)$ by ADM

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Tables (5) and (6) show a comparison between the absolute error of LTM solution and ADM solution at $\alpha = \beta = 0.95$

Table (5) Absolute error of the capacitor current $i_c(t)$

t	Error of LTM	Error of ADM
.07	0.0000327271	0.0000467271
.10	0.0000228615	0.0000227615
.20	1.37471×10^{-6}	1.47471×10^{-6}
.30	4.5572×10^{-6}	4.4572×10^{-6}
.40	5.44987×10^{-6}	5.44987×10^{-6}
.50	5.16179×10^{-6}	4.86179×10^{-6}
.60	2.91413×10^{-6}	6.51413×10^{-6}
.70	1.53798×10^{-6}	0.000050138
.80	1.61595×10^{-7}	0.000695562
.90	4.97104×10^{-6}	0.00871283

Table (6) Absolute error of the capacitor voltage $v_c(t)$

t	Error of LTM	Error of ADM
.06	0.00029488	0.00102188
.10	0.0000649332	3.32094×10^{-8}
.20	0.000578582	0.000564482
.30	0.000672405	0.000449705
.40	0.000636441	0.000639041
.50	0.000559089	0.000443789
.60	0.00110404	0.00118824
.70	0.000728657	0.00227466
.80	0.000463695	0.0248972
.90	0.000133816	0.218505

Conclusion

The fractional calculus is a powerful tool which generalizes the parallel RLC circuit. The fractional modeling introduces new parameters which provide more accurate representations of real capacitor and real inductor. The methods of ADM and LTM are suitable for solving the fractional order parallel RLC circuit.

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