Solvability in Gevrey Classes of Some Linear Functional Equations

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Abstract. In this paper, we associate to each number k > 0 a new class of linear endomorphisms of the sheaf of germs of holomorphic functions on [-1;1] and prove the solvability in the Gevrey class of some linear functional equations related to these endomorphisms. We apply the result obtained to prove the solvability in the Gevrey class G_k([-1;1]) of some linear functional equations.

Keywords and phrases: Functional equations, Gevrey classes
Mathematics Subject Classification: 30D60,65Q20

1. Introduction

The functional equations have been the subject of intensive studies because of their relation to applied and social sciences. The extreme variety of areas where functional equations are found only enhance their attractiveness. In the study of such equations there are different approaches and various research directions (cf for example [1-7]). However, in our opinion there are a few studies on their solvability in Gevrey classes.

In this paper, we associate to each number k > 0 a new class of endomorphisms of the sheaf of germs of holomorphic functions on [-1;1] and prove the solvability in the Gevrey class $G_k([-1;1])$ of linear functional equation related to these endomorphisms. We apply then the result obtained to prove the solvability in the Gevrey class G_k ([-1; 1]) of some linear functional equation.

2. Notations, Definitions and Preliminaries

In this section we summarize some basic properties and related definitions which are essential in the following discussion:

Let S be a nonempty subsets of $\mathbb{C}$ and f: S $\longrightarrow\mathbb{C}$ a bounded function $||f||_{\infty,S}$ denotes the quantity $||f||_{\infty,S} := \sup_{z\in S}|f(z)|$.

For z$\in S$ we set (z ; S):= $\inf_{u\in S}|z-u|$.

O(S) denotes the set of holomorphic functions on some neighborhood of S.

For z$\in\mathbb{C}$ and h > 0, B(z , h) is the open ball in $\mathbb{C}$ with center z and radius h.

For r > 0, k > 0,A > 0 and n$\in\mathbb{N}^*$, we set

$$[-1;1]_r := [-1;1]+B(0,r),\ [-1;1]_{k,A,n} := [-1;1]_{An^{-\frac{1}{k}}}$$

Thus we have

$[-1;1]_r = \{z\in \mathbb{C}: \varrho(z,[-1;1]) < r\}, [-1;1]_{k,A,n} = \{z\in \mathbb{C}: \varrho(z,[-1;1]) < An^{-\frac{1}{k}}\}$

Let E be a nonempty set and g :E→E a mapping. For every n∈ ℕ, $g^{\langle n\rangle}$ denotes the iterate of order n of g for the composition of mappings.

Through this paper k > 0 will be a fixed constant number. Then the Gevrey class $G_k([-1;1])$ is the set of functions f:[-1;1] → ℂ of class C^∞on [-1;1] such that there exist some constants C > 0, A > 0 verifying the following inequalities

$$||f^{(n)}||_{\infty,[-1;1]} \leq CA^n n^{n(1+\frac{1}{k})},\ n = 0, 1, 2, \dots$$

with the convention that $0^0 = 1$.

Remark 2.1

$G_k([-1;1])$ is a ℂ-vector space when it is endowed with the usual operations of addition of functions and multiplication of functions by complex constants.

Let ψ be a holomorphic function on a neighborhood of [-1;1] such that

$$\psi([-1;1])\subset [-1;1], \text{ we set } \lambda(\psi):=\sup_{n\geq 1}\left(\frac{||\psi^{(n)}||_{\infty,[-1;1]}}{n!}\right)^{\frac{1}{n}}.$$

Let us observe that λ(ψ) < ∞.

A sequence $(f_n)_{n\geq 1}$ of germs of holomorphic functions on [-1;1] is called a k-sequence if there exists a constant B>0 such that the following conditions hold for every n≥ 1

$$f_n \in O([-1;1]_{k,B,n})$$

$$||f_n||_{\infty,[-1;1]_{k,B,n}} \leq C\theta^n$$

where C > 0 and $\theta \in]0;1[$ are constants.

The following result which is a direct consequence of a theorem stated in [4], point out the link between the set of k-sequences and the Gevrey class $G_k([-1;1]$.

Theorem 2.1

A function F:[-1;1]→ ℂ belongs to the Gevrey class $G_k([-1;1])$ if and only if there exists a k-sequence $(g_n)_{n\geq 1}$ such that the function series $\sum g_n$ is uniformly convergent on [-1;1] to F.

Let $\mathcal{F}:= (\varphi_\alpha)_{\alpha\in I}$ be a family of germs of holomorphic functions on [-1;1]. We say that $\mathcal{F}$ verifies the E(k) property if there exist two constants $\sigma_0 > 0$ and $\tau_0 > 0$ depending only on $\mathcal{F}$ such that $\varphi_\alpha \in O([-1;1]_{\sigma_0}$ for all α ∈ I and for every $A\in]0;\tau_0[$ an integer M(A)≥ 1 such that

$$\varphi_\alpha([-1;1]_{k,A,n+1}) \subset [-1;1]_{k,A,n},\ n\geq M(A),\ \alpha \in I$$

The real number τ_0 is called a k-threshold for the family $\mathcal{F}$.

Remark 2.2

If the family $\mathcal{F}$ verifies the property E(k) then

$$\varphi_\alpha([-1;1]) \subset [-1;1],\ \alpha \in I$$

An endomorphism T of the sheaf O([-1;1]) of holomorphic functions on [-1;1] is said to verify the A(k) property if there exist two constants $\tau_1 > 0$ and $\rho \in]0;1[$ depending only on T such that for all A$\in]0;\tau_1]$ there exists an integer N(A) depending only on A such that

$$T(O([-1;1]_{k,A,n})) \subset [-1;1]_{k,A,n+1}\ ,\ n\geq N(A) \tag{2.1}$$

$$||T(f)||_{\infty,[-1;1]_{k,A,n+1}} \leq \rho\ ||f||_{\infty,[-1;1]_{k,A,n}}\ ,\ n\geq N(A) \tag{2.2}$$

The real number τ_1 is called a k-threshold for the endomorphism T.

Remark 2.3

Since we have for every continuous germ g on [-1;1]

$$\lim_{r\to 0, r>0} ||g||_{\infty,[-1;1]_r} = ||g||_{\infty,[-1;1]}$$

it follows from (2.1) and (2.2) that

$$\| T(f)_{|[-1;1]} ||_{\infty,[-1;1]} \leq \rho\ ||f||_{\infty,[-1;1]} \tag{2.3}$$

3. Statement of the Main Result and of its Corollary

Theorem 3.1

Let T be an endomorphism of the sheaf O([-1;1]) which verify the A(k) property.
1) T induces a unique endomorphism $\breve{T}$ of the Gevrey class $G_k([-1;1])$ such that for evey k-sequence $(f_n)_{n\geq 1}$ the sequence $(T(f_n))_{n\geq 1}$ is also a k-sequence and the following relation holds

$$\breve{T}\,((\textstyle\sum_{n=0}^{+\infty} f_n)_{|[-1;1]}) = \sum_{n=0}^{+\infty} T(f_n)_{|[-1;1]} \tag{3.1}$$

2) For every u$\in$ O([-1;1]) the linear functional equation

$$\phi - \breve{T}\,(\phi) = u \tag{3.2}$$

has a unique solution which belongs to the Gevrey class $G_k([-1;1])$.

Corollary 3.1

Let be a:=$(a_n)_{n\geq 0}$a sequence of holomorphic functions on $[-1;1]_\sigma$, and φ:= $(\varphi_n)_{n\geq 0}$ a sequence which verify the E(k) property.

Assume that

$$\sum_{n=0}^{+\infty} ||a_n||_{\infty,[-1;1]_\sigma} < 1 \tag{3.3}$$

Then the linear functional equation

$$\phi(x) - \sum_{n=0}^{+\infty} a_n(x)\phi(\varphi_n(x)) = u(x) \tag{3.4}$$

has a unique solution which furthermore belongs to the Gevrey class G_k([-1;1]).

4. Proof of the Main Result and of its Corollary

4.1 Proof of the Main Result

Let T be an endomorphism on the sheaf O([-1;1]) which verify the A(k) property. Let τ be a k-threshold for the endomorphism T.

1) Let f $\in G_k([-1;1])$. Let $(g_n)_{n\geq 1}$ and $(h_n)_{n\geq 1}$ be k-sequences such that

$$f=\sum_{n=1}^{+\infty} g_{n|_{[-1;1]}} = \sum_{n=1}^{+\infty} h_{n|_{[-1;1]}}$$

Then, there exist $A_1 > 0$, $C_1 > 0$, $0 < \theta_1 < 1$ such that the following conditions hold for all $n \geq 1$

$$g_n, h_n \in O([-1;1]_{k,A_1,n})$$

$$\max\left[||g_n||_{\infty,[-1;1]_{k,A_1,n}}, ||h_n||_{\infty,[-1;1]_{k,A_1,n}}\right] \leq C_1 .\theta_1^{n}$$

Let us set for all $n \geq 1$

$$w_n := g_{n|[-1;1]_{k,A_1,n}} - h_{n|[-1;1]_{k,A_1,n}}$$

$$\phi_n := \sum_{j=1}^{n} w_j$$

We have for all $n \geq 1$

$$T(\phi_n)_{|[-1;1]} = \sum_{j=1}^{n} T(w_j)_{|[-1;1]}$$

$$= \sum_{j=1}^{n} T(g_j)_{|[-1;1]} - \sum_{j=1}^{n} T(h_j)_{|[-1;1]}$$

Since the sequence of functions $(\phi_n)_{n\geq 1}$ is uniformly convergent on [-1;1] to the null function, it follows from the inequality (2.3) that the function series $\sum T(g_n)_{|[-1;1]}$ and $\sum T(h_n)_{|[-1;1]}$ are uniformly convergent on [-1;1] to the same function which we denote by $\breve{T}$ (f).

The mapping $\breve{T}$ is well defined and linear on the Gevrey class $G_k([-1;1])$. Furthermore, $\breve{T}$ is the unique endomorphism on $G_k([-1;1])$ which satisfies the condition (3.1).

2) Let r ∈]0;τ[such that $u \in O([-1;1]_r)$.

Consider the sequence of functions .Then there exists an integer N such that

$T\,(O([-1;1]_{k,r,n})) \subset O([-1;1]_{k,r,n+1})$, $n \geq N$

$||T(f)||_{\infty,[-1;1]_{k,r,n+1}} \leq ||f||_{\infty,[-1;1]_{k,r,n}}$, $f \in O\,([-1;1]_{k,r,n})$, $n \geq N$

It is then clear that we have for all $n \geq N$

$$T^{\langle n\rangle}(u) \in O\,([-1;1]_{k,r,n})$$

$$||T^{\langle n\rangle}(u)||_{\infty,[-1;1]_{k,r,n}} \leq ||T(u)||_{\infty,[-1;1]_{k,r,N}}\, \rho^{n-N}$$

It follows by virtue of [1] that the function series is uniformly convergent to a function $w \in G_k([-1;1])$ which is a solution of the equation (3.2).

Suppose that there exists another solution $w_1 \in G_k([-1;1])$ to equation (3.2). Then let us set $\gamma := w - w_1$. Hence, we have

$$\gamma = \breve{T}(\gamma)$$

We know that there exists a k-sequence $(\gamma_n)_{n\geq 1}$ such that

$$\gamma=\sum_{n=1}^{+\infty}\gamma_{n|_{[-1;1]}}$$

An easy induction based on the inequality (2.4) proves that we have for all $N\in\mathbb{N}$

$$\|\gamma\|_{\infty,[-1;1]}=\left\|\breve{T}^{\langle N\rangle}\gamma\right\|_{\infty,[-1;1]}$$
$$\leq\rho^N\sum_{n=1}^{+\infty}\|\gamma_{n|_{[-1;1]}}\|_{\infty,[-1;1]}$$

Since $\rho\in]0;1[$ it follows that $\gamma=0$, thence $w=w_1$. Thence the equation (3.2) has a unique solution in the Gevrey class $G_k([-1;1])$. □

4.2 Proof of the Corollary

Let $f\in O([-1;1])$, then there exists $\alpha\in]0;\tau_1]$ such that $f\in O([-1;1]_\alpha)$ where $\tau_1\in]0;\sigma]$ is a k-threshold of the sequence φ. Then there exists $M\in\mathbb{N}^*$ such that

$$\varphi_p([-1;1]_{k,\alpha,n+1})\subset[-1;1]]_{k,\alpha,n},\ n\geq M,\ p\in\mathbb{N}$$

It follows that

$$f\circ\varphi_p\in O([-1;1]]_{k,\alpha,M+1},\ p\in\mathbb{N}$$
$$|a_p(z)f[\varphi_p(z)]|\leq\|a_p\|_{\infty,[-1;1]_\sigma}\|f\|_{\infty,[-1;1]_{k,\alpha,M+1}},z\in[-1;1]_{k,\alpha,M+1}$$

Hence from the condition (3.2) entails that the function series $\sum a_p(f\circ\varphi_p)$ is uniformly convergent on $[-1;1]_{k,\alpha,M+1}$.

Hence, if we set

$$T_1(f):=\sum_{p=1}^{+\infty}a_p(f\circ\varphi_p)$$

we define an endomorphism T_1 of O([-1;1]).

Let us prove that T_1 verifies the A(k) property. Let $A\in]0;\tau_1]$, there exists an integer L(A) depending only on A satisfying the following properties

$$\varphi_p([-1;1]_{k,A,n+1})\subset[-1;1]]_{k,A,n},\ n\geq L(A),\ p\in\mathbb{N}$$

Let $v\in O([-1;1]_{k,A,n})$ where n is an integer such that $n\geq L(A)$. Then

$$v\circ\varphi_p\in O([-1;1]]_{k,A,n+1},\ p\in\mathbb{N}$$
$$\|a_p\left(v\circ\varphi_p\right)\|_{\infty,[-1;1]_{k,A,n+1}}\leq\|a_p\|_{\infty,[-1;1]_\sigma}\|v\|_{\infty,[-1;1]_{k,A,n}}$$

It follows that

$$T_1(v)\in O([-1;1]]_{k,A,n+1}$$

$$\|T_1(v)\|_{\infty,[-1;1]_{k,A,n+1}}\leq\sum_{p=0}^{+\infty}\|a_p\|_{\infty,[-1;1]_\sigma}\|v\|_{\infty,[-1;1]_{k,A,n}}$$

It follows from the inequality (3.3) that the endomorphism T_1 satisfy the A(k) property.

Hence, according to the main result, it follows that the linear functional equation (3.4) has for every $u\in O([-1;1])$ a unique solution which, furthermore, belongs to the Gevrey class $G_k([-1;1])$.□

5. Some Examples

We need first to prove some useful propositions.

Proposition 5.1

Let ψ be a holomorphic function on a neighborhood of [-1;1] such that

$$\psi([-1;1]\subset[-1;1]$$

Assume that $\lambda(\psi) \leq 1$. Then the function ψ satisfies the E(1) property.

Proof

Let $\sigma > 0$ be such that $\psi \in O([-1;1]_\sigma)$.

Let $A\in]0;\min(1,\sigma)[$,$p \in \mathbb{N}^*$and $z \in[-1;1]_{k,A,p+1}$. Let $\hat{z}$ the closest point of [-1;1] to z. We have the following inequalities

$$\varrho(\psi(z),[-1;1]) \leq |\psi(z)-\psi(\hat{z})| \leq \sum_{n=1}^{+\infty} \frac{|\psi^{(n)}(\hat{z})|}{n!}|z-\hat{z}|^n$$

$$\leq \sum_{n=1}^{+\infty} \varrho(z,[-1;1])^n \leq \frac{A}{p+1-A} < \frac{A}{p}$$

It follows that

$$\psi([-1;1]_{k,A,p+1}) \subset [-1;1]_{k,A,p} \text{ , } p \in \mathbb{N}^*$$

Hence, the function ψ has the E(1) property.□

Proposition 5.2

Let g be an entire function such that

$$g([-1;1]) \subset [-1;1] \text{ and } \lambda(g) \leq 1$$

1-Let $(P_n)_{n\geq1}$be a sequence of holomorphic functions on $[-1;1]_\sigma$ such that we have for every n$\in \mathbb{N}^*$

$$P_n([-1;1]) \subset [-1;1] \text{ and } \lambda(P_n) \leq 1$$

The sequence$(g\circ P_n)_{n\geq1}$ verifies the E(1) property.

2-The sequence of functions $(g_n)_{n\geq1}$defined by the formula

$$g_n(z):=g^{\langle n\rangle}(\frac{z}{2^{n-1}}), z \in \mathbb{C}$$

verifies the E(1) property.

Proof

1-For every n$\in\mathbb{N}^*$ we have $g\circ P_n \in O([-1;1]_\sigma)$.

Let $A \in]0;\min(\frac{1}{2},\sigma)[$,$p \in \mathbb{N}^*$ and $z \in[-1;1]_{1,A,p+1}$. Let $\hat{z}$ be the closest point of [-1;1] to z. We have the following inequalities

$$\varrho(g\circ P_n(z),[-1;1]) \leq | g\circ P_n (z)- g\circ P_n (\hat{z})|$$

$$\leq \sum_{j=1}^{+\infty} \frac{|g^{(j)}(P_n(\hat{z}))|}{j!} |P_n(z) - P_n(\hat{z})|^j$$
$$\leq \sum_{j=1}^{+\infty} \left[\sum_{m=1}^{+\infty} \frac{|P_n{}^{(m)}(\hat{z})|}{m!} |z-\hat{z}|^m\right]^j$$
$$\leq \sum_{j=1}^{+\infty} \left[\sum_{m=1}^{+\infty} \varrho(z,[-1;1])^m\right]^j \leq \frac{A}{p+1-2A} < \frac{A}{p}$$

It follows that

$g \circ P_n([-1;1]_{1,A,p+1}) \subset [-1;1]_{1,A,p}$, $p \in \mathbb{N}^*, n \in \mathbb{N}^*$

Hence, the sequence $(g \circ P_n)_{n\geq 1}$ verifies the E(1) property.

2-For every $n \in \mathbb{N}^*$, g_n is an entire function such that $g_n([-1;1]) \subset [-1;1]$.
Let us show by induction that $\lambda(g_n) \leq 1$, for every $n \in \mathbb{N}^*$.
We have $\lambda(g_1) = \lambda(g) \leq 1$.
Assume that $\lambda(g_n) \leq 1$ for a certain $n \geq 1$. Then by virtue of Faa-di-Bruno formula we have for every $x \in [-1;1]$ and $p \in \mathbb{N}^*$

$$\frac{|g_n{}^{(p)}(x)|}{p!} \leq \frac{1}{2^p} \sum_{j_1+2j_2+\cdots+pj_p=p} \frac{(j_1+j_2+\cdots+j_p)!}{j_1!j_2!\ldots j_p!} \frac{|g^{(j_1+j_2+\cdots+j_p)}(g_n(\frac{x}{2}))|}{(j_1+j_2+\cdots+j_p)!} \prod_{s=1}^{p} \left(\frac{|g_n{}^{(s)}(\frac{x}{2})|}{s!}\right)^{j_s}$$

$$\leq \frac{1}{2^p} \sum_{j_1+2j_2+\cdots+pj_p=p} \frac{(j_1+j_2+\cdots+j_p)!}{j_1!j_2!\ldots j_p!} \leq 1$$

It follows that $\lambda(g_{n+1}) \leq 1$.
Hence we have

$$\lambda(g_n) \leq 1,\ n \in \mathbb{N}^*$$

Hence, according to the previous part of this proposition, the sequence $(g_n)_{n\geq 1}$ verifies the E(1) property.□

Example-1

Direct computations show that $\lambda(\sin) = 1$. Hence, the function sin verifies the E(1) property. It follows that the linear functional equation

$$\phi(x) - \frac{1}{2}\phi(\sin x) = x \tag{5.1}$$

has a unique solution which belongs to the Gevrey class $G_1([-1;1])$.

Example-2

Let $(P_n)_{n\geq 1}$ be a sequence of holomorphic functions on $[-1;1]_\sigma$ such that we have for every $n \in \mathbb{N}^*$

$$P_n([-1;1]) \subset [-1;1] \text{ and } \lambda(P_n) \leq 1$$

Let g be an entire function such that

$$g([-1;1]) \subset [-1;1] \text{ and } \lambda(g) \leq 1$$

Let $a:=(a_n)_{n\geq 0}$ be a sequence of holomorphic functions on $[-1;1]_\sigma$ such that

$$\sum_{n=0}^{+\infty} ||a_n||_{\infty,[-1;1]_\sigma} < 1$$

Then the functional equation

$$\phi(x) - \sum_{n=0}^{+\infty} a_n(x)\phi(g(P_n(x))) = u(x) \tag{5.2}$$

has for every u ∈ O([-1;1]) a unique solution which belongs to the Gevrey class $G_1([-1;1])$.

Example-3

Let $(\alpha_n)_{n\geq 1}$ be a sequence of real numbers. f_n denotes for every $n \in \mathbb{N}^*$ the entire function and is defined by

$$f_n(z) := \sin(z-\alpha_n),\ z \in \mathbb{C}$$

Direct computations show that the linear functional equation

$$\phi(x) - \sum_{n=0}^{+\infty} \frac{x^2}{2^{n+1}(x^2+1)} \phi(\sin(\sin(x-\alpha_n))) = u(x) \tag{5.3}$$

has for every u ∈ O([-1;1]) a unique solution which belongs to the Gevrey class $G_1([-1;1])$.

Example-4

According to the previous proposition and to the fact that $\lambda(\sin) = 1$, it follows that the sequence of functions $(g_n)_{n\geq 1}$ defined by

$$g_n(z) := \sin^{\langle n \rangle}\left(\frac{z}{2^{n-1}}\right),\ z \in \mathbb{C}$$

verifies the E(1) property.

Hence the linear functional equation

$$\phi(x) - \sum_{n=0}^{+\infty} \frac{\cos(\varepsilon_n x)}{2^{n+1}} \phi\left(\sin^{\langle n \rangle}\left(\frac{z}{2^{n-1}}\right)\right) = u(x) \tag{5.4}$$

has for every u ∈O([-1;1]) and every bounded sequence $(\varepsilon_n)_{n\geq 1}$ of real numbers a unique solution which belongs to the Gevrey class $G_1([-1;1])$.

Acknowledgments. We would express our profound gratitude to the referee and to Professor Salah Badawi Ahmed Doma for their kindness and their great support.

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