Construction of LPA codes correcting simultaneously random and P-burst block errors

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Abstract. Linear partition Arihant (LPA) codes [2] have been introduced by the author in [2] and a study of error correcting/detecting capabilities of these code was made with respect to the random block errors. Also, the concept of P-burst errors has been formulated by the author in [4] to study clustered block errors that occur during the process of communication. In this paper, we take up the problem of simultaneous correction of random block errors and P-burst errors in LPA spaces and obtain a construction upper bound on the number of parity check digits required for the same.

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1. Introduction

P-burst error correcting/detecting codes [4] are suitable for correcting/detecting block errors which do not occur independently but are clustered over a particular block length. These type of errors occur in many situations. One important and practical situation is in which the message is disturbed over a particular block length together with occasional disturbances, thus, creating simultaneously P-bursts as well as random block errors. Such a situation arise, e.g., in semiconductor memory systems where the memory is highly vulnerable to clustered block errors due to bombardment of strong radioactive particles such as cosmic particles on RAM chips and occasional/random block errors result from decay of RAM chips. Therefore, in actual communication/storage while it is important to consider correction of P-bursts, care must be taken to correct random block errors of upto a specified Arihant weight irrespective of the block position where they occur. Keeping this in view, in this paper, we obtain a construction upper bound on the number of parity check digits required for LPA codes correcting simultaneously random and P-burst block errors.

2. Preliminaries

Let q, n be positive integers with $q > 1$. Let $\mathbf{Z}_q$ be the ring of integers modulo q. Let $\mathbf{Z}_q^n$ be the set of all n-tuples over $\mathbf{Z}_q$. Then $\mathbf{Z}_q^n$ is a module over $\mathbf{Z}_q$. For q prime, $\mathbf{Z}_q$ becomes a field and $\mathbf{Z}_q^n$ becomes a vector space over $\mathbf{Z}_q$. A partition P of the positive integer n is defined as

$$\begin{aligned} P \ &: \ n = n_1 + n_2 \cdots + n_s \quad \text{where} \\ & \quad 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s, s \geq 1. \end{aligned}$$

The partition P is denoted as

$$P : n = [n_1][n_2] \cdots [n_s].$$

In the case, when

$$\begin{aligned} P : n \ &= \ \underbrace{[m_1] \cdots [m_1]}_{l_1\text{- copies}} \underbrace{[m_2] \cdots [m_2]}_{l_2\text{- copies}} \\ & \quad \cdots \underbrace{[m_r] \cdots [m_r]}_{l_r\text{- copies}} , \end{aligned}$$

we write

$$P : n = [m_1]^{l_1} [m_2]^{l_2} \cdots [m_r]^{l_r},$$

where $m_1 < m_2 < \cdots < m_r$.

Given a partition $P : n = [n_1][n_2] \cdots [n_s]$ of the positive integer n, the module space $\mathbf{Z}_q^n$ over $\mathbf{Z}_q$ can be viewed as a direct sum

$$\begin{aligned} \mathbf{Z}_q^n \ &= \ \mathbf{Z}_q^{n_1} \oplus \mathbf{Z}_q^{n_2} \oplus \cdots \oplus \mathbf{Z}_q^{ns}, \\ & \quad \text{or} \\ V \ &= \ V_1 \oplus V_2 \oplus \cdots V_s, \end{aligned}$$

where $V = \mathbf{Z}_q^n$ and $V_i = \mathbf{Z}_q^{n_1}$ for all $1 \leq i \leq s$.

Consequently, each vector $v \in \mathbf{Z}_q^n$ can be uniquely written as $v = (v_1, v_2, \cdots, v_s)$ where $v_i \in V_i = \mathbf{Z}_q^{n_i}$ for all $1 \leq i \leq s$.

Here $v_i (1 \leq i \leq s)$ is called the i^{th} *block of block size* n_i of the vector v. Further, we define the modular value $|a|$ of an element $a \in \mathbf{Z}_q$ by

$$|a| = \begin{cases} a & \text{if} \quad 0 \leq \ a \leq q/2 \\ q-a & \text{if} \quad q/2 < a \leq q-1. \end{cases}$$

We note that non-zero modular value $|a|$ can be obtained by two different elements viz. a and $q-a$ of $\mathbf{Z}_q$ provided $\{q$ is odd$\}$ or $\{q$ is even and $a \neq [q/2]\}$ i.e.

$$|a| = |q-a| \quad \text{if} \quad \begin{cases} q \ \text{ is odd} \\ \ \text{or} \\ q \text{ is even and } \ a \neq q/2. \end{cases}$$

If q is even and $a = [q/2]$ or if $a = 0$, then $|a|$ is obtained in only one way viz. $|a| = a$. Thus there may be one or two equivalent values of $|a|$ which we shall refer to as repetitive equivalent values of a. The number of repetitive equivalent values of a will be denoted by e_a where

$$e_a = \begin{cases} 1 & \text{if } \ \{ q \ \text{ is even and } \ a = [q/2]\} \ \text{ or } \{a = 0\} \\ 2 & \text{if } \ \{ q \ \text{ is odd and } a \neq 0\} \ \text{ or } \{q \text{ is even, } a \neq 0 \ \text{ and } a \neq [q/2]\}. \end{cases}$$

3. Definitions and notations

We begin with the discussion of LPA codes [2]:

Let n, q be positive integers with $q > 1$. Let $P : n = [n_1][n_2] \cdots [n_s]$ be a partition of n. We define *Arihant metric* on $\mathbf{Z}_q^n$ corresponding to the partition P as follows:

Let $v = (v_1, v_2, \cdots, v_s) \in \mathbf{Z}_q^n = \mathbf{Z}_q^{n_1} \oplus \mathbf{Z}_q^{n_2} \oplus \cdots \oplus \mathbf{Z}_q^{n_s}$. The Arihant weight of the i^{th} block $v_i \in \mathbf{Z}_q^{n_i} (1 \leq i \leq s)$ of the vector v corresponding to the partition P of n is defined as

$$w_A^P(v_i) = \max_{j=1}^{n_i} |v_j^{(i)}|$$

where

$$v_i \quad = \quad (v_1^{(i)}, v_2^{(i)}, \cdots, v_{n_i}^{(i)}) \in \mathbf{Z}_q^{n_i}.$$

Thus the Arihant weight of a block is the maximum modular value amongst all its components. Then the Arihant weight of the vector $v = (v_1, v_2, \cdots, v_s) \in \mathbf{Z}_q^{n_1} \oplus \mathbf{Z}_q^{n_2} \oplus \cdots \oplus \mathbf{Z}_q^{n_s}$ corresponding to the partition P is defined as the sum of Arihant weights of all its blocks i.e.

$$W_A^P(v) = \sum_{i=1}^{s} w_A^P(v_i).$$

For any $u = (u_1, u_2, \cdots, u_s)$ and $v = (v_1, v_2, \cdots, v_s) \in \mathbf{Z}_q^n = \mathbf{Z}_q^{n_1} \oplus \mathbf{Z}_q^{n_2} \oplus \cdots \oplus \mathbf{Z}_q^{n_s}$, we define the Arihant distance (or Arihant metric) $d_A^P(u, v)$ between u and v as

$$d_A^P(u, v) = w_A^P(u - v).$$

Then d_A^P is a metric on $\mathbf{Z}_q^n = \mathbf{Z}_q^{n_1} \oplus \cdots \oplus \mathbf{Z}_q^{n_s}$.

If the partition P is clear from the context, we shall denote Arihant weight by w_A and Arihant metric by d_A only.

Definition 3.1 [2]. A linear partition Arihant (LPA) code corresponding to the partition $P : n = [n_1] \cdots [n_s]$ is a $\mathbf{Z}_q$-submodule of $\mathbf{Z}_q^n = \mathbf{Z}_q^{n_1} \oplus \mathbf{Z}_q^{n_2} \oplus \cdots \oplus \mathbf{Z}_q^{n_s}$ equipped with the Arihant metric and is denoted as $[n, k, d_A; P]$ or $[n, k; P]$ code where

$$k = \text{rank}_{\mathbf{Z}_q}(V),$$

and

$$\begin{aligned} d_A &= d_A(V) \\ &= \text{minimum Arihant distance} \\ &\quad \textit{of } V \\ &= \min\{d_A(u, u') \mid u, u' \in V, \\ &\quad u \neq u'\}. \end{aligned}$$

Remark 3.2.

1. For $P : n = [1]^n$, the linear partition Arihant codes reduce to the classical Lee weight codes [6,7]. For this partition, the Arihant distance and Arihant weight reduce to classical Lee distance and Lee weight respectively.

2. For $q = 2, 3$, the linear partition Arihant codes reduce to the linear error-block codes [1] and the Arihant metric reduces to the π-metric introduced by Feng et al. [1].

3. In general, we have

$$\begin{aligned} \pi\text{-metric} &\leq \text{Arihant metric} \\ &\leq \text{Lee metric}, \end{aligned}$$

or equivalently

$$\begin{aligned} \pi\text{-weight} &\leq \text{Arihant weight} \\ &\leq \text{Lee weight}. \end{aligned}$$

We now define P-burst in $\mathbf{Z}_q^n = \mathbf{Z}_q^{n_1} \oplus \mathbf{Z}_q^{n_2} \oplus \cdots \oplus \mathbf{Z}_q^{n_s}$ as follows [4].

Definition 3.3 [4]. Let n be a positive integers and $P : n = [n_1][n_2] \cdots [n_s]$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ be a partition of n. A P-burst of block length b $(1 \leq b \leq s)$ is a vector $v = (v_1, v_2, \cdots, v_s) \in \mathbf{Z}_q^n = \bigoplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ such that all the non-zero blocks in v are confined to some b consecutive block positions, the first and last of which are non-zero.

Definition 3.4 [4]. A P-burst of block length b or less $(1 \leq b \leq s)$ is a P-burst of block length t where $1 \leq t \leq b \leq s$.

Throughout this paper, we shall use the following notations:

1. $[x]$ = The largest integer less than or equal to x.

2. $\lceil x \rceil$ = The smallest integer greater than or equal to x.

3. Q_i=The sum of repetitive equivalent values up to i i.e.,

$$Q_i = e_0 + e_1 + \cdots + e_i$$

where e_i denotes the repetitive equivalent value of i.

4. Construction upper bound for LPA codes correcting simultaneously random and P-burst block errors

In this section, we obtain a sufficient bound on the number of parity checks required for an LPA code that correct all random block errors of Arihant weight t or less ($t \geq 1$) simultaneously with all P burst errors of block length b or less ($b \geq 1$) with Arihant weight w or less ($w \geq t$).

To prove the result, we need the following.

Let $A_{t,q}^{(n_1,n_2,\cdots,n_s)}$ [5] denote the number of all n-vectors corresponding to the partition $P : n = [n_1][n_2]\cdots[n_s]$ with $1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ having Arihant weight t over $\mathbf{Z}_q$. Then $A_{t,q}^{(n_1,n_2,\cdots,n_s)}$ is given by :

$$A_{t,q}^{(n_1,n_2,\cdots,n_s)} = \sum_{r=(r_{ij})} \left(\prod_{i=1}^{s} \prod_{j=0}^{[q/2]} ((Q_j)^{n_i} - (Q_{j-1})^{n_i})^{r_{ij}} \right), \tag{1}$$

where $r = (r_{ij})(1 \leq i \leq s, 0 \leq j \leq [q/2]$ satisfies

(i) for a fixed $i(1 \leq i \leq s), r_{ij} = 1$ for exactly one value of $j(0 \leq j \leq [q/2])$ and 0 elsewhere; and

(ii)

$$\sum_{i=1}^{s} \sum_{j=0}^{[q/2]} j r_{ij} = t. \tag{2}$$

Again if $V_{t,q}^{(n_1,n_2,\cdots,n_s)}$ denote the number of all n-vector corresponding to the partition $P : n = [n_1][n_2]\cdots[n_s], 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ having Arihant weight t or less over $\mathbf{Z}_q$. Then $V_{t,q}^{(n_1,n_2,\cdots,n_s)}$ is given by

$$V_{t,q}^{(n_1,n_2,\cdots,n_s)} = \sum_{j=0}^{t} A_{t,q}^{(n_1,n_2,\cdots,n_s)} \tag{3}$$

We now give a definition for linear combination of vectors having Arihant weight w.

Definition 4.1. A linear combination of vectors $u_1, u_2, \cdots, u_r$ given by

$$\lambda_1.u_1 + \lambda_2.u_2 + \cdots + \lambda_r.u_r,$$

where $\lambda_i = (\lambda_1^{(i)}, \lambda_2^{(i)} \cdots \lambda_{n_i}^{(i)}), u_i = (u_1^{(i)}, u_2^{(i)}, \cdots, u_{n_i}^{(i)}) \in \mathbf{Z}_q^{n_i}$ for all $1 \leq i \leq r$ and (.) denote the usual dot product of vectors, is called a linear combination of Arihant weight w if

$$\max_{a=1}^{n_1}|\lambda_a^{(1)}| + \max_{b=1}^{n_2} + |\lambda_b^{(2)}| + \cdots + \max_{l=1}^{n_r}|\lambda_l^{(r)}| = w.$$

The following lemma enumerates the number of Arihant weighted P-bursts in block coding:

Lemma 4.2. The number of P bursts of block length b or less with Arihant weight $g \geq 1$ in the space of all $(n_1+n_2+\cdots+n_{j-1})$-block vectors over the ring $\mathbf{Z}_q$ is given by

$$\begin{aligned} C_q(b, \sum_{i=1}^{j-1} n_i, g) &= C_q(1, \sum_{i=1}^{j-1} n_i, g) + \sum_{m=2}^{b}\sum_{r=1}^{j-m}\Big(\sum_{\substack{2\leq\lambda_1+\lambda_2\leq g \\ \lambda_1,\lambda_2\geq 1}} ((Q_{\lambda_1})^{n_r} - (Q_{\lambda_{1-1}})^{n_r}) \times \\ &\quad ((Q_{\lambda_2})^{n_{r+m-1}} - (Q_{\lambda_2-1})^{n_{r+m-1}}) A_{g-\lambda_1-\lambda_2,q}^{(n_{r+1},n_{r+2},\cdots,n_{r+m-2})}\Big), \end{aligned} \quad (4)$$

where

$$C_q(1, \sum_{i=1}^{j-1} n_i, g) = \begin{cases} \sum_{i=1}^{j-1}((Q_g)^{n_i} - (Q_{g-1})^{n_i}) & \text{if } g \leq [q/2], \\ 0 \quad \text{if } g > [q/2], \end{cases}$$

and $A_{g-\lambda_1-\lambda_2,q}^{(n_{r+1},n_{r+2},\cdots,n_{r+m-2})}$ is given by (1) satisfying (2).

Proof. There are two cases:

Case 1. When $b = 1$.

In this case, the number of P-bursts of block length 1 with Arihant weight g in the space of all $n_1 + n_2 + \cdots + n_{j-1}$-tuples over $\mathbf{Z}_q$ is given by

$$C_q(1, \sum_{i=1}^{j-1} n_i, g) = \begin{cases} \sum_{i=1}^{j-1}((Q_g)^{n_i} - (Q_{g-1})^{n_i}) & \text{if } g \leq [q/2], \\ 0 \quad \text{if } g > [q/2]. \end{cases} \quad (5)$$

Case 2. When $b \geq 2$.

Consider a P-burst of block length m and Arihant weight g where $2 \leq m \leq b \leq j-1$ and $g \geq 1$. Such a P-burst can have first $(j-m)$ block positions as the starting block positions. Suppose the P-burst starts at the r^{th} block $(1 \leq r \leq j-m)$ and suppose that the Arihant weights of the starting and ending blocks are $\lambda_1(\neq 0)$ and $\lambda_2(\neq 0)$ resp. Then the number of choices for the starting r^{th} block and ending $(r+m-1)^{th}$ block together is given by

$$((Q_{\lambda_1})^{n_r} - (Q_{\lambda_1 - 1})^{n_r})((Q_{\lambda_2})^{n_{r+m-1}} - (Q_{\lambda_2 - 1})^{n_{r+m-1}}). \tag{6}$$

The remaining $(m-2)$ blocks viz $(r+1)^{th}, (r+2)^{th}, \cdots, (r+m-2)^{th}$ blocks of the P-burst should make up a sum of Arihant weight $g - \lambda_1 - \lambda_2$ so that the total Arihant weight of the P-burst becomes equal to g. The number of ways in which these $(m-2)$ blocks can be filled is given by

$$A_{g-\lambda_1-\lambda_2,q}^{(n_{r+1},n_{r+2},\cdots,n_{r+m-2})}, \tag{7}$$

where $A_{g-\lambda_1-\lambda_2,q}^{(n_{r+1},n_{r+2},\cdots,n_{r+m-2})}$ is given by (1) satisfying (2).

The total number of P-bursts of Arihant weight g and block length m starting from the r^{th} block is obtained by multiplying (6) and (7) and then summing the resulting product for different values of $\lambda_1' s$ and $\lambda_2' s$ satisfying $\lambda_1 \geq 1, \lambda_2 \geq 1,\ 2 \leq \lambda_1 + \lambda_2 \leq g$ and is given by

$$\sum_{\substack{2 \leq \lambda_1+\lambda_2 \leq g \\ \lambda_1,\lambda_2 \geq 1}} ((Q_{\lambda_1})^{n_r} - (Q_{\lambda_{1-1}})^{n_r})((Q_{\lambda_2})^{n_{r+m-1}} - (Q_{\lambda_2-1})^{n_{r+m-1}}) \times A_{g-\lambda_1-\lambda_2,q}^{(n_{r+1},n_{r+2},\cdots,n_{r+m-2})}. \tag{8}$$

Since r can take values from 1 to $j-m$ and m can take values from 2 to b, therefore, summing (8) for different values for m and r gives the number of P-bursts of block length varying from 2 to b and having Arihant weight g and is given by

$$\sum_{m=2}^{b}\sum_{r=1}^{j-m}\Big(\sum_{\substack{2 \leq \lambda_1+\lambda_2 \leq g \\ \lambda_1,\lambda_2 \geq 1}} ((Q_{\lambda_1})^{n_r} - (Q_{\lambda_{1-1}}^{n_r}) \times ((Q_{\lambda_2})^{n_{r+m-1}} - (Q_{\lambda_2-1})^{n_{r+m-1}})A_{g-\lambda_1-\lambda_2,q}^{(n_{r+1},n_{r+2},\cdots,n_{r+m-2})}\Big). \tag{9}$$

The result now follows by adding (5) and (9). □

Remark 4.3.(i) The number of all P-bursts in $\mathbf{Z}_q^n = \bigoplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ of block length b or less ($b \leq s$) with Arihant weight w or less is given by

$$C_q^*(b, \sum_{i=1}^{s} n_i, w) = 1 + \sum_{g=1}^{w} C_q(b, \sum_{i=1}^{s} n_i, g).$$

(ii) The number of all P-bursts in $\mathbf{Z}_q^n = \bigoplus_{1=1}^{s} \mathbf{Z}_q^{n_i}$ of block length b or less ($b \leq s$) having Arihant weight lying between w_1 and w_2 is given by

$$C_q^*(b, \sum_{i=1}^{s} n_i, w_1, w_2) = \sum_{g=w_1}^{w_2} C_q(b, \sum_{i=1}^{s} n_i, g).$$

We now prove the sufficient bound which is infact a construction upper bound on the number of parity check digits required for LPA codes correcting simultaneously random and P-burst block errors.

Theorem 4.4. *Let n be a positive integer and $P : n = [n_1][n_2] \cdots [n_s], 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ be a partition of n. Let t, w and b be positive integers such that $1 \leq t \leq w \leq b[q/2]$ and $2 \leq b \leq s$. Then a sufficient condition for the existence of an $[n, k; P]$ LPA code over $\mathbf{Z}_q$ (q prime) that correct all random block errors of Arihant weight t or less and all P-burst of block length b or less with Arihant weight w or less is given by*

$$q^{n-k} \geq V_{2t,q}^{(n_1,n_2,\cdots,n_s)} + \sum_{\lambda=1}^{[q/2]} \Big((Q_\lambda)^{n_s} - (Q_\lambda)^{n_s} \Big) \times D, \tag{10}$$

where

$$\begin{aligned} D = & \left(\sum_{\substack{p_1,p_2: \\ p_1+p_2=2t+1-\lambda}}^{t+w-\lambda} C_q(b, \sum_{i=1}^{s-1} n_i, p_1) A_{p_2,q}^{(n_1,n_2,\cdots,n_{s-1})} \right) + \\ & \left(V_{w-\lambda,q}^{(n_{s-b+1},n_{s-b+2},\cdots,n_{s-1})} - V_{t-\lambda,q}^{(n_{s-b+1},n_{s-b+2},\cdots,n_{s-1})} \right) \times \\ & A_{t,q}^{(n_1,n_2,\cdots,n_{s-b})} + \left(C_q^*(b, \sum_{i=1}^{s-b} n_i, t+1, w) \right) \times \end{aligned}$$

$$\left(V_{w-\lambda,q}^{(n_{s-b+1},n_{s-b+2},\cdots,n_{s-1})} - (V_{t-\lambda,q}^{(n_{s-b+1},n_{s-b+2},\cdots,n_{s-1})}\right) +$$
$$\left(C_q^*(b-1, \sum_{s-b+1}^{s-1} n_i, 2t+2-\lambda, 2w-\lambda)\right) +$$
$$\sum_{k=1}^{b-1} \sum_{\theta=0}^{(min[q/2],w-1)} \sum_{r_{1\theta},r_{2\theta},r_{3\theta}} \left((Q_\theta^{n_{s-2b+k+1}} - (Q_{\theta-1})^{n_{s-2b+k+1}}\right) \times$$
$$A_{r_{1\theta},q}^{(n_{s-2b+k+1},\cdots,n_{s-b})} A_{r_{2\theta},q}^{(n_{s-b+1},\cdots,n_{s-b+k})} A_{r_{3\theta},q}^{(n_{s-b+k+1},\cdots,n_{s-1})},$$

and

$$2 \leq p_1 \leq w; 0 \leq p_2 \leq t-\lambda; 1 \leq \theta + r_{1\theta} \leq w-1;$$
$$1 \leq +r_{2\theta} \leq 2w-1-\lambda; 2t+2-\lambda \leq \theta + r_{1\theta} + r_{2\theta} + r_{3\theta} \leq 2w-\lambda;$$
$$0 \leq r_{3\theta} \leq w-\lambda; r_{2\theta} + r_{3\theta} \geq t+1-\lambda;$$
$$\theta + r_{1\theta} + r_{2\theta} \geq t+1.$$

(**Note.** The functions C_q and C_q^* are defined in Lemma 4.2. and Remark 4.3 respectively.)

Proof. The existence of such a code will be proved by constructing an appropriate $(n-k) \times n$ parity check matrix H for the desired LPA code. Suppose we have chosen first $(j-1)$ blocks viz. $H_1, H_2, \cdots, H_{j-1}$ of block sizes $n_1, n_2, \cdots, n_{j-1}$ resp. The j^{th} block H_j of size n_j can be added to the parity check matrix H if the conditions in the following three cases are fulfilled:

Case 1. Since the LPA code is to correct all combinations of Arihant weight t or less, therefore, the minimum Arihant distance of the code must be at least $2t+1$. Thus, the block $H_j = (h_1^{(j)}, h_2^{(j)} \cdots h_{nj}^{(j)})$ to be added to the parity check matrix H can be any set of n_j column vectors of length $n-k$ satisfying

$$\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_j.H_j \neq 0,$$

where $\lambda_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{Z}_q^{n_i}$ for all $1 \leq i \leq j$ and $1 \leq w_A(\lambda_1) + w_A(\lambda_2) + \cdots + w_A(\lambda_j) = \max_{a=1}^{n_1}|\lambda_a^{(1)}| + \max_{b=1}^{n_2}|\lambda_b^{(2)}| + \cdots + \max_{l=1}^{n_j}|\lambda_l^{(j)}| \leq 2t.$

The number of such linear combinations including the vector of all-zeros is given by

$$V_{2t,q}^{(n_1,n_2,\cdots,n_j)} \tag{11}$$

where $V_{2t,q}^{(n_1,n_2,\cdots,n_j)}$ is given by (3).

Case 2. Secondly, since the code is required to correct random block errors of Arihant weight t or less simultaneously with all P-burst errors of block length b or less with Arihant weight w or less, therefore, the syndrome of any two error patterns, one random of Arihant weight t or less, another a P-burst of block length b or less with Arihant weight w or less must not be the same except when both error patterns are same. This will imply that no codeword is the sum or difference of a random block error of Arihant weight t or less and a P-burst of block length b or less with Arihant weight w or less (except when the two errors are same), i.e. any random block error of Arihant weight t or less is not in the same coset of the standard array as that of any P-burst error pattern of block length b or less with Arihant weight w or less.

For this condition to be satisfied, again we have two subcases:

Subcase (i) When j^{th} block is included in the random block error of Arihant weight t or less.

In this case. we have

$$(\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_j.H_j) + (\beta_c.H_c + \beta_d.H_d + \cdots + \beta_l.H_l) \neq 0, \quad (12)$$

where $\beta_c.H_c + \beta_d.H_d + \cdots + \beta_l.H_l(\beta_i \in \mathbf{Z}_q^{n_i}$ for all $1 \leq i \leq l)$ is any linear combination of Arihant weight w or less from b or fewer consecutive blocks taken from previous chosen $j-1$ blocks viz. $H_1, H_2, \cdots, H_{j-1}$ and $\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_j.H_j(\lambda_i \in \mathbf{Z}_q^{n_i}$ for all $1 \leq i \leq j-1, \lambda_j \in \mathbf{Z}_q^{n_j}/0\}$ is any linear combination of Arihant weight t or less of the first previously chosen $(j-1)$ blocks viz. $H_1, H_2, \cdots, H_{j-1}$ and the j^{th} block H_j to be added. Equivalently, we can say that $\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_{j-1}.H_{j-1}$ is any linear combination of Arihant weight $t-\lambda$ or less of the previously chosen first $(j-1)$ blocks where $1 \leq \lambda = w_A(\lambda_j) = \max_{i=1}^{n_j}|\lambda_i^{(j)}| \leq [q/2]$.

The condition (12) assures that the syndrome of any block error pattern of Arihant weight t or less is not equal to that of any P-burst of block length b or less with Arihant weight w or less in all cases except when subcase (ii) occurs.

Subcase (ii). When j^{th} block is included in the P-burst of block length b or less and the patterns of Arihant weight exactly equal

to "t" are selected from the first $j-b$ blocks.

In this case, we have

$$(\alpha_j.H_j + \alpha_{j-1}.H_{j-1} + \cdots + \alpha_{j-b+1}.H_{j-b+1}) + \\ (\gamma_1.H_1 + \gamma_2.H_2 + \cdots + \gamma_{j-b}.H_{j-b}) \neq 0, \qquad (13)$$

where $\alpha_i \in \mathbf{Z}_q^{n_i}$ for all $j-b+1 \leq i \leq j, \alpha_j \neq (0,0,\cdots,0)$; $\gamma_l \in \mathbf{Z}_q^{n_l}$ for all $1 \leq l \leq j-b$ and $1 \leq w_A(\gamma_1) + w_A(\gamma_2) + \cdots + w_A(\gamma_{j-b}) = t$.

Here $\gamma_1.H_1 + \gamma_2.H_2 + \cdots + \gamma_{j-b}.H_{j-b}$ is any linear combination of Arihant weight exactly equal to "t" from the first $j-b$ blocks viz $H_1, H_2, \cdots, H_{j-b}$ and $\alpha_j.H_j + \alpha_{j-1}.H_{j-1} + \cdots + \alpha_{j-b+1}.H_{j-b+1}$is any linear combination of Arihant weight w or less of the j^{th} block to be added and the immediately preceding $b-1$ blocks viz. $H_{j-1}, H_{j-2}, \cdots, H_{j-b+1}$. Equivalently, we can say $\alpha_{j-1}.H_1 + \cdots + \alpha_{j-b+1}.H_{j-b+1}$ is any linear combination of Arihant weight $w-\lambda$ or less of the immediately preceding $(b-1)$ blocks where $1 \leq \lambda = w_A(\alpha_j) = w_A(\lambda_j) = \max_{i=1}^{n_j} |\lambda_i^{(j)}| \leq [q/2]$.

We now enumerate all possible linear combinations occuring in (12) and (13). Since all possible linear combinations of Arihant weight $2t$ or less (including the weight of the j^{th} block) are included in (11), therfore we choose coefficients in (12) such that

$$\begin{aligned} 2t+1 &\leq \lambda + \sum_{i=1}^{j-1} \max_{u=1}^{n_i} |\lambda_u^{(i)}| + \sum_{v=c}^{l} \max_{y=1}^{n_v} |\beta_y^{(v)}| \leq t+w \\ \Rightarrow 2t+1-\lambda &\leq \sum_{i=1}^{j-1} \max_{u=1}^{n_i} |\lambda_u^{(i)}| + \sum_{v=c}^{l} \max_{y=1}^{n_v} |\beta_y^{(v)}| \leq t+w-\lambda, \end{aligned} \qquad (14)$$

where $1 \leq \lambda = \max_{i=1}^{n_j} |\lambda_i^{(j)}| \leq [q/2]$.

Also,

$$t+1-\lambda \leq w_A(\alpha_{j-1}) + \cdots + w_A(\alpha_{j-b+1}) \leq w-\lambda. \qquad (15)$$

In order to do so, let

$$w_A(\beta_c) + w_A(\beta_d) + \cdots + w_A(\beta_l) = \sum_{v=c}^{l} \max_{y=1}^{n_v} |\beta_y^{(v)}| = p_1,$$

and

$$w_A(\lambda_1) + w_A(\lambda_2) + \cdots + w_A(\lambda_{j-1}) = \sum_{i=1}^{j-1} \max_{u=1}^{n_i} |\lambda_u^{(i)}| = p_2,$$

then we have

$$2t + 1 - \lambda \leq p_1 + p_2 \leq t + w - \lambda, \tag{16}$$

where $2 \leq p_1 \leq w, 0 \leq p_2 \leq t - \lambda$.

Now, $\beta_i's$ $(c \leq i \leq l)$ which form a P-burst of block length b or less with Arihant weight p_1 in a block vector of size $n_1 + n_2 + \cdots + n_{j-1}$ can be selected (using Lemma 4.2) in

$$C_q(b, \sum_{i=1}^{j-1} n_i, p_1) \qquad \text{ways.} \tag{17}$$

Also, the $\lambda_m's$ $(1 \leq m \leq j-1)$ can be selected in

$$A_{p_2,q}^{(n_1,n_2,\cdots,n_{j-1})} \qquad \text{ways.} \tag{18}$$

Thus the total number of linear combinations occuring in in (12) is given by

$$\sum_{\lambda=1}^{[q/2]} ((Q_\lambda)^{n_j} - (Q_{\lambda-1})^{n_j}) \sum_{\substack{p_1,p_2: \\ p_1+p_2=2t+1-\lambda}}^{t+w-\lambda} C_q(b, \sum_{i=1}^{j-1} n_i, p_1) A_{p_2,q}^{(n_1,n_2,\cdots,n_{j-1})}, \tag{19}$$

where p_1, p_2 satisfy (16).

Further the $\alpha_i's$ $(j - b + 1 \leq i \leq j - 1)$ satisfying (15) can be chosen in

$$\sum_{\lambda=1}^{[q/2]} ((Q_\lambda)^{n_j} - (Q_{\lambda-1})^{n_j}) \times \left(V_{w-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})} - V_{t-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})} \right) \quad \text{ways.} \tag{20}$$

Finally, $\gamma_e (1 \leq e \leq j - b)$ can be chosen in

$$A_{t,q}^{(n_1,n_2,\cdots,n_{j-b})} \qquad \text{ways.} \tag{21}$$

Thus, the total number of linear combinations occuring in (13) are given by

$$\sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{n_j}-(Q_{\lambda-1})^{n_j})\times \left(V_{w-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}-V_{t-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}\right)\times A_{t,q}^{(n,n_2,\cdots,n_{j-b})} \quad \text{ways.} \tag{22}$$

Therefore, the total number of linear combination arising out of case (2) is obtained by adding (20) and (22) and is given by

$$\sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{n_j}-(Q_{\lambda-1})^{n_j})\Bigg(\Bigg(\sum_{\substack{p_1,p_2:\\ p_1+p_2=2t+1-\lambda}}^{t+w-\lambda} C_q(b,\sum_{i=1}^{j-1}n_i,p_1)\times A_{p_2,q}^{(n_1,n_2,\cdots,n_{j-1})}\Bigg)+\left(V_{w-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}-V_{t-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}\right)\times A_{t,q}^{(n_1,n_2,\cdots,n_{j-1})}\Bigg), \tag{23}$$

where p_1, p_2 satisfy the inequality (16).

Case 3. Lastly, we are to exclude the possibility of the same syndrome of any block error pattern each of which is a P-burst of block length b or less with Arihant weight w or less. For this case, the sum of any two linear combinations each of Arihant weight w or less, the one involving j^{th} block H_j and the immediately preceding $(b-1)$ blocks viz. $H_{j-1}, H_{j-2}, \cdots, H_{j-b+1}$ and the second involving b (or fewer) consecutive blocks amongst the first $(j-1)$ blcoks chosen so far should not to be zero i.e.

$$(\eta_j.H_j+\eta_{j-1}.H_{j-1}+\cdots+\eta_{j-b+1}.H_{j-b+1}) +(\delta_{i_1}.H_{i_1}+\delta_{i_2}.H_{i_2}+\cdots+\delta_{i_b}.H_{i_b})\neq 0, \tag{24}$$

where

$$\begin{aligned}
&\{H_{i_1}, Hi_2,\cdots,Hi_b\}\subseteq\{H_1,H_2,\cdots,H_{j-1}\};\\
&\eta_i\in \mathbf{Z}_q^{n_i} \text{ for all } j-b+1\le i\le j,\ \eta_j\neq(0,0,\cdots,0);\\
&\delta_{i_l}\in \mathbf{Z}_q^{n_i} \text{ for all } 1\le l\le b;\\
&t+1\le w_A(\eta_j)+w_A(n_{j-1})+\cdots+w_A(n_{j-b+1})\le w;\\
&t+1\le w_A(\eta_{i_1})+w_A(\delta_{i_2})+\cdots+w_A(\delta_{i_b})\le w.
\end{aligned} \tag{25}$$

To compute the number of all possible linear combinations occuring in (24) for all possible choices of η_i $(j-b+1 \leq i \leq j; \eta_j \neq (0,0,\cdots,0)$ and δ_{i_l} $(1 \leq l \leq b)$, we analyze the situation in three different subcases:

Subcase (i). When $\delta_{i_l}(1 \leq l \leq b)$ in (24) are taken from the first $(j-b)$ blocks.

It ic clear from (25) that

$$t+1 \leq \sum_{l=1}^{b} w_A(\delta_{i_l}) \leq w, \tag{26}$$

and

$$t+1-\lambda \leq \sum_{l=j-b+1}^{j-1} w_A(\eta_i) \leq w-\lambda, \tag{27}$$

where $1 \leq \lambda = w_A(\eta_j) \leq [q/2]$.

The number of $\eta_i(j-b+1 \leq i \leq j-1)$ satisfying (27) is given in (20) whereas $\delta_{i_l}(1 \leq l \leq b)$ which form a P-burst of block length b or less with Arihant weight lying between $(t+1)$ and w in an $(n_1+n_2+\cdots+n_{j-b})$-block vector can be chosen in

$$C_q^*(b, \sum_{i=1}^{j-b} n_i, t+1, w) = \sum_{g=t+1}^{w} C_q(b, \sum_{i=1}^{j-b} n_i, g) \quad \text{ways}, \tag{28}$$

where $C_q(b, \sum_{i=1}^{j-b} n_i, g)$ is given in (4) in Lemma 4.2.

Subcase (ii). When $\delta_{i_l}(1 \leq l \leq b)$ in (24) are taken from the immediately preceding $(b-1)$ blocks.

In this case, the number of additional ways in which η_i $(j-b+1 \leq i \leq j-1)$, $\eta_j \neq (0,o,\cdots,0)$ and $\delta_{i_l}(1 \leq l \leq b)$ can be selected is given by

$$\sum_{\lambda=1}^{[q/2]} ((Q_\lambda)^{nj} - (Q_{\lambda-1})^{nj})(C_q^*(b-1, \sum_{i=j-b+1}^{j-1} n_i, 2t+2-\lambda, 2w-\lambda). \tag{29}$$

Subcase (iii). When $\delta_{i_l}(1 \leq l \leq b)$ in (24) are neither completly confined to first $(j-b)$ blocks nor to the last $(b-1)$ blocks.

In this case, the $\delta_{i_l}(1 \leq l \leq b)$ are selcetd from the blocks $H_{j-2b+2}, H_{j-2b+3}, \cdots, H_{j-1}$ in such a way that not all are taken either from $H_{j-2b+2}, H_{j-2b+3}, \cdots, H_{j-b}$ or from $H_{j-b+1}, H_{j-b+2}, \cdots, H_{j-1}$. Let us suppose that the P-burst starts from the $(j-2b+k+1)^{th}$ block position which may continue upto $(j-b+k)^{th}$ block position $(1 \leq k \leq b-1)$. Also, let the Arihant weight of the $(j-2b+k+1)^{th}$ block be θ where $1 \leq \theta \leq \min([q/2], w-1)$. The total number of choices for selecting the components of $(j-2b+k+1)^{th}$ block is given by

$$(Q_\theta)^{n_{j-2b+k+1}} - (Q_{\theta-1})^{n_{j-2b+k+1}}.$$

Our objective is to select non-zero blcoks from the $(j-2b+k+1)^{th}, (j-2b+k+2)^{th}, \cdots, (j-b)^{th}, (j-b+1)^{th}, \cdots, (j-b+k)^{th}, (j-b+k+1)^{th}, \cdots, (j-b)^{th}$ blocks having sum of their Arihant weight w or less. or this, let us have linear combinations of Arihant weight $r_{1\theta}$ of blocks of columns from the $(j-2b+k+2)^{th}, \cdots, (j-b)^{th}$ blocks; linear combination of Arihant weight $r_{2\theta}$ of columns from the $(j-b+1)^{th} \cdots, (j-b+k)^{th}$ blocks and lienar combinations of Arihant weight $r_{3\theta}$ of columns from the $(j-b+k+1)^{th}, \cdots, (j-)^{th}$ blocks. The total number of choices of the linear combinations occuring in (24) arising out of subcase (iii) turns out to be

$$\begin{aligned}
&\sum_{\lambda=1}^{[q/2]} \Big((Q_\lambda)^{nj} - (Q_{\lambda-1})^{nj} \Big) \\
&\Big(\sum_{k=1}^{b-1} \sum_{\theta=1}^{(min[q/2], w-1)} \sum_{r_{1\theta}, r_{2\theta}, r_{3\theta}} \\
&((Q_\theta)^{n_{j-2b+k+1}} - (Q_{\theta-1})^{n_{j-2b+k+1}}) A_{r_{1\theta,q}}^{(n_{j-2b+k+1}, \cdots, n_{j-b})} \times \\
&A_{r_{2\theta,q}}^{(n_{j-b+1}, \cdots, n_{j-b+k})} A_{r_{3\theta,q}}^{(n_{j-b+k+1}, \cdots, n_{j-1})},
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&1 \leq \theta + r_{1\theta} \leq w-1; \\
&1 \leq +r_{2\theta} \leq 2w-1-\lambda; \\
&0 \leq +r_{3\theta} \leq w-\lambda; \\
&r_{2\theta} + r_{3\theta} \geq t+1-\lambda; \\
&\theta + r_{1\theta} + r_{2\theta} \geq t+1 \\
&2t+2-\lambda \leq \theta + r_{1\theta} + r_{2\theta} + r_{3\theta} \leq 2w-\lambda.
\end{aligned}$$

Thus the total number of possible distinct linear combinations arising out of Case 3 are given by

$$
\begin{aligned}
& (20)\times(28)+(29)+(30)\\
= & \left(C_q^*(b,\sum_{i=1}^{j-b}n_i,t+1,w)\right)\left(\sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{nj}-(Q_{\lambda-1})^{nj})\times\right.\\
& \left.\left(V_{w-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}-(V_{t-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}\right)\right)+\\
& \sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{nj}-(Q_{\lambda-1})^{nj})\left(C_q^*(b-1,\sum_{i=j-b+1}^{j-1}n_i,2t+2-\lambda,2w-\lambda)\right)+\\
& \sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{nj}-(Q_{\lambda-1})^{nj})\Big(\sum_{k=1}^{b-1}\sum_{\theta=1}^{(min[q/2],w-1)}\sum_{r_{1\theta},r_{2\theta},r_{3\theta}}\\
& ((Q_\theta)^{n_{j-2b+k+1}}-(Q_{\theta-1})^{n_{j-2b+k+1}})\,A_{r_{1\theta},q}^{(n_{j-2b+k+1},\cdots,n_{j-b})}\times\\
& A_{r_{2\theta},q}^{(n_{j-b+1},\cdots,n_{j-b+k})}A_{r_{3\theta},q}^{(n_{j-b+k+1},\cdots,n_{j-1})}. \qquad (31)
\end{aligned}
$$

Therefore, the total number of possible distinct linear combinations arising out of all the three cases including the pattern of all zeros is given by

$$
\begin{aligned}
& (11)+(23)+(31)\\
= & V_{2t,q}^{(n_1,n_2,\cdots,n_j)}+\\
& \sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{n_j}-(Q_{\lambda-1})^{n_j})\Big(\Big(\sum_{\substack{p_1,p_2:\\ p_1+p_2=2t+1-\lambda}}^{t+w-\lambda}C_q(b,\sum_{i=1}^{j-1}n_i,p_1)\times\\
& \left.A_{p_2,q}^{(n_1,n_2,\cdots,n_{j-1})}\right)+\left(V_{w-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}-V_{t-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}\right)\times\\
& \left.A_{t,q}^{(n_1,n_2,\cdots,n_{j-1})}\right)+\left(C_q^*(b,\sum_{i=1}^{j-b}n_i,t+1,w)\right)\left(\sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{nj}-(Q_{\lambda-1})^{nj})\times\right.\\
& \left.\left(V_{w-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}-(V_{t-\lambda,q}^{(n_{j-b+1},n_{j-b+2},\cdots,n_{j-1})}\right)\right)+\\
& \sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{nj}-(Q_{\lambda-1})^{nj})\left(C_q^*(b-1,\sum_{i=j-b+1}^{j-1}n_i,2t+2-\lambda,2w-\lambda)\right)+\\
& \sum_{\lambda=1}^{[q/2]}((Q_\lambda)^{nj}-(Q_{\lambda-1})^{nj})\Big(\sum_{k=1}^{b-1}\sum_{\theta=1}^{(min[q/2],w-1)}\sum_{r_{1\theta},r_{2\theta},r_{3\theta}}
\end{aligned}
$$

$$\begin{aligned}
&((Q_\theta)^{n_{j-2b+k+1}} - (Q_{\theta-1})^{n_{j-2b+k+1}})\, A_{r_{1\theta,q}}^{(n_{j-2b+k+1},\cdots,n_{j-b})} \times \\
&A_{r_{2\theta,q}}^{(n_{j-b+1},\cdots,n_{j-b+k})} A_{r_{3\theta,q}}^{(n_{j-b+k+1},\cdots,n_{j-1})} \\
= \; & L \quad \text{(say)}.
\end{aligned}$$

Hence the j^{th} block H_j of block size n_j can be added to H if the number of $(n-k)$-tuples i.e. q^{n-k} is at least as large as L i.e. if

$$q^{n-k} \geq L. \tag{32}$$

For the existence of an $[n, k; P]$ LPA code where $n = n_1 + n_2 = \cdots + n_s$, the inequality (32) must hold for $j = s$ so that it is possible to add upto s^{th} block H_s to form an $(n-k) \times n$ block matrix and we get (10).

6. Conclusion

In this paper, we have obtained a construction upper bound for LPA codes over $\mathbf{Z}_q$ correcting random block errors and P-burst errors simultaneously. The bound has been obtained by an algorithmic procedure by way of constructing a suitable parity check matrix for the code.

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