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# Non-binary $l_{t_i/n_i}$-ispotty byte error control $\theta$-codes

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**Abstract.** Irregular-spotty-byte error control codes devised by the author[3-5] are suitable for binary semiconductor memories with binary arithmatic where a memory chip consists of irregular bytes of not necessarily of same length. However, direct storage and processing of non-binary numbers in base 10, base 8 or base 16 is possible with the instant invention. Keeping this in view, in this paper, we formulate the concept of non-binary $l_{t_i/n_i}$-ispotty byte error control codes suitable for non-binary storage and computing using RAM chips of irregular bytes.

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**Keywords:** ispotty byte codes, ispotty distance

## 1. Introduction

Irregular-spotty-byte error control codes devised by the author [3-5] are suitable for correcting/detecting errors in binary semiconductor memory systems having ibyte-organized memory chips where ibytes are memory bytes not necessarily of the same length. A memory chip under this configuration consists of "$s$" ibytes where length of the $j^{th}$ ibyte is $n_j (n \geq 1, 1 \leq j \leq s)$. These semiconductor memory systems are useful in computer and other communication systems such as mobile systems, aircrafts, satellites etc. The multiple errors arising in these semiconductor memories while being exposed to strong electromegnatic waves, radioactive particles or energetic cosmic particles are isopotty-byte errors or $t_i/n_i$-errors[3-5]. The study of ispotty-byte error control codes [3-5] has been made with respect to the ispotty distance induced from the classical Hamming distance. However, now a days, the memory elements may be programmed to store and process non-binary digits. Direct storage and processing of numbers in base 10, base 8 or base 16 etc is possible with the instant invention. As a result, higher storage densities and non-binary operation possible with multistage memory elements through the instant computing methods provides an opportunity to vastly improve the speed and efficiency of computation relative to conventional binary computing machines [9]. Keeping this in view, there is a need for non-binary ispotty byte errors control codes. Since the binary ispotty distance is induced from the classical Hamming distance and we know that the classical non-binary Lee distance is stronger than the Hamming distance since in the case of Hamming distance, any digital change in one place is a single errors, no matter what the magnitude of change is, whereas in the case of Lee distance, a digital change of $\pm t$ in one place contributes "$t$" errors. Therefore, in this paper, we present a model of non-binary ispotty-byte error control codes viz. $l_{t_i/n_i}$-codes equipped with a non-binary ispotty distance induced from the classical Lee distance [7]. We present various bounds on the parameters of non-binary $l_{t_i/n_i}$-ispotty byte error control codes capable of detecting/correcting non-binary $l_{t_i/n_i}$-errors.

Throughout this paper, $[x]$ denotes the greatest integer less than or equal to $x$ and $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

## 2. Definitions and notations

Let $q, n$ be positive integers with $q \geq 2$. Let $\mathbf{Z}_q$ be the ring of integers modulo $q$. Let $\mathbf{Z}_q^n$ be the set of all $n$-tuples over $\mathbf{Z}_q$. Then $\mathbf{Z}_q^n$ is a module over $\mathbf{Z}_q$. Let $V$ be a submodule of the module $\mathbf{Z}_q^n$ over $\mathbf{Z}_q$. For $q$ prime, $\mathbf{Z}_q$ becomes a field and $\mathbf{Z}_q^n$ becomes a vector space and subspace respectively over $\mathbf{Z}_q$. A partition $P$ of the positive integer $n$ is defined as

$$P : n \;=\; n_1 + n_2 + \cdots + n_s, 1 \leq n_1 \leq n_2 \leq n_3 \cdots\cdots \geq n_s \leq 1, \quad s \geq 1,$$

and is denoted as

$$\begin{aligned}
&P : n = [n_1][n_2]\cdots[n_s] \\
&= [m_1]^{l_1}[m_2]^{l_2}\cdots[m_r]^{l_r} \text{ if} \\
&n_1 = n_2 = \cdots = n_{l_1} = m_1, \\
&n_{l_1+1} = n_{l_1+2} = \cdots = n_{l_1+l_2} = m_2, \\
&\vdots
\end{aligned}$$

$$\vdots$$
$$\vdots$$
$$n_{l_1+l_2+\cdots+l_{r-1}+1}+n_{l_1+l_2+\cdots+l_{r-1}+2}$$
$$+\cdots+n_{l_1+l_2+\cdots+l_r}=m_r.$$

Then we can write the field $\mathbf{Z}_q^n$ as

$$\mathbf{Z}_q^n=\mathbf{Z}_q^{n_1}\oplus\mathbf{Z}_q^{n_2}\oplus\cdots\oplus\mathbf{Z}_q^{n_s}.$$

Each vector $v\in\mathbf{Z}_q^n$ can be uniquely written as $v=(v_1,v_2,\cdots,v_s)$ where $v_i\in V_i=\mathbf{Z}_q^{n_i}$ for all $1\le i\le s$ and is called the $i^{th}$ *irregular-byte* or simply $i^{th}$ *i-byte of* $v$. We call the partition $P$ as *primary-partition* or *irregular-byte-partition*. Further, let $1\le t\le n$ be a positive integer and let $P': t=[t_1][t_2]\cdots[t_s]$ be a partition of $t$ where $1\le t_i\le n_i$ for all $1\le i\le s$. Then $P'$ is called as *"secondary-partition"* or *"error-partition"*. Note that the secondary partition depends upon primary partition. The number $n$ is called the *primary number* and $t$ is called the *secondary number*.

Further, we define the modular value or Lee weight $|a|$ (or $w_l(a)$) of an element $a\in\mathbf{Z}_q$ by

$$w_l(a)=|a|=\begin{cases} a & \text{if} \quad 0\le\ a\le q/2,\\ q-a & \text{if} \quad q/2<a\le q-1.\end{cases}$$

We note that non-zero modular value $|a|$ can be obtained by two different ways viz. $a$ and $q-a$ of $\mathbf{Z}_q$ provided $\{q$ is odd$\}$ or $\{q$ is even and $a\neq[q/2]\}$ i.e.

$$|a|=|q-a| \quad \text{if} \quad \begin{cases} q \ \text{ is odd}\\ \ \text{or}\\ q \text{ is even and } \ a\neq q/2.\end{cases}$$

If $q$ is even and $a=[q/2]$ or if $a=0$, then $|a|$ is obtained in only one way viz. $|a|=a$. Thus there may be one or two equivalent values of $|a|$ which we shall refer to as repetitive equivalent values of $a$. The number of repetitive equivalent values of $a$ will be denoted by $e_a$ where

$$e_a=\begin{cases} 1 & \text{if } \{\ q \ \text{ is even and } \ a=[q/2]\} \ \text{ or } \{a=0\}\\ 2 & \text{if } \{\ q \ \text{ is odd and } a\neq 0\} \ \text{ or } \{q \text{ is even, } a\neq 0 \ \text{ and } a\neq[q/2]\}.\end{cases}$$

**Definition 2.1.** Let $n$ and $t$ be the positive integers with $1 \leq t \leq n$. Let $P$ and $P'$ be the primary and secondary partitions corresponding to $n$ and $t$ respectively given by

$$\begin{array}{rcl} P : n & = & [n_1][n_2]\cdots[n_s], \\ \text{and} & & \\ P' : t & = & [t_1][t_2]\cdots[t_s], \end{array}$$

where $1 \leq t_i \leq n_i$ for all $1 \leq i \leq s$.

Let $u$ be a vector in $\mathbf{Z}_q^n = \oplus_{i=1}\mathbf{Z}_q^{n_i}$ given by

$$u = (u_1, u_2, \cdots, u_s)$$

where $u_i \in \mathbf{Z}_q^{n_i}$ for all $i$ is the $i^{th}$ i-byte of $u$ of size $n_i$. We define the *non-binary ispotty weight* (or simply $l_{t_i/n_i}$-measure) of $u$ corresponding to the primary-partition $P$ and secondary-partition $P'$ as

$$w_{l_{t_i/n_i}}^{(P,P')} = \sum_{i=1}^{s} \left\lceil \frac{w_l(u_i)}{t_i} \right\rceil,$$

where $w_l(u_i)$ is the Lee weight of the $i^{th}$ ibyte $u_i$ of $u$ of size $n_i$.

**Definition 2.2.** The *non-binary ispotty distance* (or equivalently $l_{t_i/n_i}$-distance) between two vectors $u = (u_1, u_2, \cdots, u_s)$ and $v = (v_1, v_2, \cdots v_s)$ in $\mathbf{Z}_q^n = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$ is given by

$$\begin{aligned} d_{l_{t_i/n_i}}^{(P,P')}(u, v) &= w_{l_{t_i/n_i}}^{(P,P')}(u - v) \\ &= \sum_{i=1}^{s} \left\lceil \frac{d_l(u_i, v_i)}{t_i} \right\rceil, \end{aligned}$$

where $d_l(u_i, v_i)$ is the Lee distance between the $i^{th}$ ibytes $u_i$ and $v_i$ of $u$ and $v$ respectively. Then $l_{t_i/n_i}$-distance is a metric function on $\mathbf{Z}_q^n = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$ (proven in Theorem 3.1)

**Note.** We will call denote the non-binary ispotty weight and non-binary ispotty distance viz. $w_{l_{t_i/n_i}}^{(P,P')}$ and $d_{l_{t_i/n_i}}^{(P,P')}$ by $w_\theta$ and $d_\theta$ respectively when the primary-partition $P$ and secondary-partition $P'$ are clear from the context.

**Definition 2.3.** Let $t$ and $n$ be positive integers with $1 \leq t \leq n$. Let $V \subseteq \mathbf{Z}_q^n = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$ be a $\mathbf{Z}_q$-submodule of $\mathbf{Z}_q^n = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$ equipped with the non-binary ispotty metric $d_\theta$ corresponding to the primary-partition $P$ of $n$ and secondary-partition $P'$ of $t$. Then $V$ is called a *non-binary ispotty byte error control code* (or simply *$\theta$-code)* and is denoted by $[n, k, d_\theta; P, P']$ where $P : n = [n_1][n_2]\cdots[n_s]$ is the irregular-byte partition, $P' : t = [t_1][t_2]\cdots[t_s]$, $1 \leq t_i \leq n_i$ is the error-partition, $k = \dim_{\mathbf{Z}_q} V$ and $d_\theta =$ minimum $\theta$ distance$= \min\limits_{\substack{x,y\in V\\ x\neq y}} d_\theta(x, y)$.

## 3. Properties of $l_{t_i/n_i}$-codes

We begin by proving that $l_{t_i/n_i}$-distance defined in Section 2 is indeed a metric function.

**Theorem 3.1.** *The $\theta$-distance $d_\theta$ corresponding to the primary-partition $P : n = [n_1][n_2]\cdots[n_s]$ and secondary-partition $P' : t = [t_1][t_2]\cdots[t_s]$, $1 \leq t_i \leq n_i$ for all $i \leq i \leq s$ is metric function on $\mathbf{Z}_q^n = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$.*

**Proof.** Let $x = (x_1, x_2, \cdots, x_s), y = (y_1, y_2, \cdots, y_s)$ and $z = (z_1, z_2, \cdots z_s)$ be arbitrary vectors in $\mathbf{Z}_q^n = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$ where $x_i, y_i, z_i \in \mathbf{Z}_q^{n_i}$ for all $i$. Then

(i) Clearly $d_\theta(x, y) > 0$ if $x \neq y$ and $d_\theta(x, y) = 0$ if $x = y$.

(ii) $d_\theta(x, y) = d_\theta(y, x)$

(iii) Since $d_l(x_i, y_i) \leq d_l(x_i, z_i) + d_l(z_i, y_i)$,
therefore,

$$\frac{d_l(x_i, y_i)}{t_i} \leq \frac{d_l(x_i, z_i)}{t_i} + \frac{d_l(z_i, y_i)}{t_i},$$

which further gives

$$\begin{aligned} \left\lceil \frac{d_l(x_i, y_i)}{t_i} \right\rceil &\leq \left\lceil \frac{d_l(x_i, z_i)}{t_i} + \frac{d_l(z_i, y_i)}{t_i} \right\rceil \\ &\leq \left\lceil \frac{d_l(x_i, z_i)}{t_i} \right\rceil + \left\lceil \frac{d_l(z_i, y_i)}{t_i} \right\rceil. \end{aligned} \tag{1}$$

Taking summation from $i = 1$ to $s$ in (1) gives

$$d_\theta(x, y) \leq d_\theta(x, z) + d_\theta(z, y)$$

Hence $d_\theta$ is a metric function on $\mathbf{Z}_q^N = \oplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$. □

**Remarks 3.2.**

(i) Let $t'$, $s$ and $b$ be positive integers with $1 \leq t' \leq b$. Taking $n = bs, t = t's$, $n_i = b$ and $t_i = t'$ for all $i$, then $\theta$-distance (weight) reduces to the $t'/b$-distance (weight) introduced by the author [5].

(ii) If $t_i = 1$ for all $1 \leq i \leq s$, then $w_\theta(x)$ for $x = (x_1, x_2, \cdots, x_s) \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ is expressed as

$$\begin{aligned} w_\theta(x) &= \sum_{i=1}^{s} \left\lceil \frac{w_l(x_i)}{1} \right\rceil \\ &= \sum_{i=1}^{s} w_l(x_i) \\ &= \text{Lee weight of } x. \end{aligned}$$

(iii) If $t_i = n_i$ for all $1 \leq i \leq s$ i.e. when secondary partition $P'$ is equal to the primary partition $P$, then $w_\theta(x)$ for $x = (x_1, \cdots, x_s) \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ is expressed as

$$\begin{aligned} w_\theta(x) &= \sum_{i=1}^{s} \left\lceil \frac{w_l(x_i)}{n_i} \right\rceil \\ &= \alpha\text{-weight of } x \text{ [6]}. \end{aligned}$$

(iv) If $q = 2, 3$ then $\theta$-weight ($\theta$-distance) coincides with the binary ispotty-weight (ispotty-distance) introduced by the author [3].

(v) Let $\lambda_i = \left\lceil \dfrac{n_i[q/2]}{t_i} \right\rceil$ for all $1 \leq i \leq s$. Then $\lambda_i$ is the maximum $\theta$-measure of an error pattern that can occur in the $i^{th}$ byte of size $n_i$. Let $\hat{\lambda} = \sum_{i=1}^{s} \lambda_i$. Then $\hat{\lambda}$ is the maximum $\theta$-measure of an error pattern that can occur in a word $x = (x_1, x_2, \cdots x_s) \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$.

(vi) Let $\theta_Z(x)$ be the total number of (erroneous) ibytes in a word $x \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ having errors of $\theta$-measure equal to "$Z$" where $Z = 0, 1, 2, \cdots, \lambda$; $\lambda = \max_{i=1}^{s}\{\lambda_i\}$ and $\lambda_i' s$ are as given in (v).

Let

$$\begin{aligned}\sigma &= \theta_1(x) + \theta_2(x) \cdots + \theta_\lambda(x) \\ &= \text{total number of erroneous ibytes in } x.\end{aligned}$$

The total number of ibytes in the word $x$ is expressed as

$$\begin{aligned}s &= \sigma + \theta_0(x) \\ &= \theta_0(x) + \theta_1(x) + \cdots + \theta_\lambda(x).\end{aligned}$$

Using these functions $\theta_Z{}'s$, the $\theta$-measure of $x \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ is expressed as

$$w_\theta(x) = \theta_1(x) + 2\theta_2(x) + \cdots + \lambda\theta_\lambda(x),$$

where

$$\lambda = \max_{i=1}^{s}\{\lambda_i\} = \max_{i=1}^{s}\Big\{\Big\lceil \frac{n_i[q/2]}{t_i}\Big\rceil\Big\}.$$

We now give a definition of linear combination of vectors (in $\mathbf{Z}_q^{n_i}$) of Lee weight $w_l$.

**Definition 3.3.** A linear combination of vectors $u_1^{(i)}, u_2^{(i)}, \cdots, u_{n_i}^{(i)}$ (in $\mathbf{Z}_q^{n_i}$) given by

$$\alpha_1 u_1^{(i)} + \alpha_2 u_2^{(i)} + \cdots + \alpha_{n_i} u_{n_i}^{(i)},$$

where $\alpha_j \in \mathbf{Z}_q, u_j^{(i)} \in \mathbf{Z}_q^{n_i}$ for all $1 \leq j \leq n_i$ is called a *linear combination of Lee weight* $w_l$ if the Lee weight of the $n_i$-vector $(\alpha_1, \alpha_2, \cdots, \alpha_{n_i}) \in \mathbf{Z}_q^{n_i}$ is $w_l$.

Now we give a necessary and sufficient condition for the $\theta$-distance of an ispotty byte code to be equal to $d$.

**Theorem 3.4.** *Let $P : n = [n_1][n_2]\cdots[n_s]$ and $P' : t = [t_1][t_2]\cdots[t_s]$ be the primary and secondary partitions corresponding to primary-number $n$ and secondary-number $t$ respectively where $1 \leq t_i \leq n_i$ for all $1 \leq i \leq s$. Let $H = [H_1, H_2, \cdots, H_s]$ be an $r \times n$ parity check matrix of an $[n, n-r; P, P']$ $\theta$-code $V$ over $\mathbf{F}_q$ where $H_i (1 \leq i \leq s)$ is the $i^{th}$ $r \times n_i$ $q$-ary submatrix of $H$. Then the minimum $\theta$-distance of code $V$ is "$d$" iff the following two conditions hold:*

(i) *$xH^T \neq 0$ for all $x = (x_1, x_2, \cdots x_s) \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ with $w_\theta(x) \leq d-1$, and*

(ii) *there exists $u = (u_1, u_2, \cdots, u_s) \in \oplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ satisfying*

$$uH^T = 0 \quad \text{and } w_\theta(x) = d.$$

**Proof.** The proof follows from the fact that syndrome of a received word $x$ having errors of $\theta$-measure $(d-1)$ or less is not equal to zero and also there exists a word of $u$ of $\theta$-measure "$d$" whose syndrome is zero. □

**Theorem 3.5.** *A linear $[n, n-r, d; P, P']$ $\theta$-code over $\mathbf{Z}_q$ corresponding to the primary partition $P : n = [n_1][n_2] \cdots [n_s]$ and secondary partition $P' : t = [t_1][t_2] \cdots [t_s]$ and having minimum $\theta$-distance "$d$" requires at least $\left[\frac{d-1}{[q/2]}\right] t'$ parity check digits where $t' = \max_{i=1}^{s}\{t_i\}$ or equivalently*

$$r \geq \left[\frac{d-1}{[q/2]}\right] t'. \tag{2}$$

**Proof.** By Theorem 3.2, every linear combination of Lee weight $(d-1)t_i$ or less of columns of the submatrix $H_i$ of $H$ are linearly independent over $\mathbf{Z}_q$. Since maximum modular value of an element in $\mathbf{Z}_q$ is $[q/2]$, therefore, equivalently we can say that every set of $\left[\frac{d-1}{[q/2]}\right] t_i$ or fewer columns of the submatrix $H_i$ of $H$ is linearly independent over $\mathbf{Z}_q$ and hence

$$r \geq \left[\frac{d-1}{[q/2]}\right] t_i \quad \text{for all } 1 \leq i \leq s$$

or equivalently

$$r \geq \left[\frac{d-1}{[q/2]}\right] t' \quad \text{where } t' = \max_{i=1}^{s}\{t_i\}.$$

□

**Definition 3.6.** An $[n, n-r; P, P']$ $\theta$-code $V$ is called an MDS code if equality holds in (2).

**Example 3.7.** Let $q = 5, n = t = 3$. Let $P : P' : 3 = [1][2]$ be the primary and secondary partitions. Then $n_1 = t_1 = 1$ and $n_2 = t_2 = 2$ and $s = 2$. Let $V$ be a $[3, 1, d_\theta; P, P']$ $\theta$-code over $\mathbf{Z}_5$ with parity check matrix

$$\begin{aligned} H &= (H_1 \vdots H_2) \\ &= \begin{pmatrix} 1 & \vdots & 1 & 0 \\ 1 & \vdots & 0 & 1 \end{pmatrix}. \end{aligned}$$

The generator matrix of the code $V$ is given by

$$\begin{aligned} G &= (G_1 \vdots G_2) \\ &= \begin{pmatrix} 4 & \vdots & 4 & 1 \end{pmatrix}. \end{aligned}$$

The five codewords of the code $V$ are

$$\begin{aligned} v_0 &= (0 \vdots 0\ 0), w_\theta(v_0) = 0; \\ v_1 &= (4 \vdots 4\ 1), w_\theta(v_1) = 2; \\ v_2 &= (3 \vdots 3\ 2), w_\theta(v_2) = 3; \\ v_3 &= (2 \vdots 2\ 3), w_\theta(v_3) = 4; \\ v_4 &= (1 \vdots 1\ 4), w_\theta(v_4) = 2. \end{aligned}$$

The minimum $\theta$-weight and hence minimum $\theta$-distance of the code $V$ is 2. Thus $V$ is an MDS code as the parameters of code $V$ satisfy the relation

$$r = \left[\frac{d-1}{[q/2]}\right] t'$$

as $r = 2; t' = max\{1, 2\} = 2, q = 5$ and $d = 2$.

Now, we obtain the Hamming sphere packing bound for $\theta$-codes. To obtain the desired bound, we need to find $V_{d,q}^{(t_i/n_1, t_2/n_2, \cdots, t_i/n_s)}$ where $V_{d,q}^{(t_i/n_1, t_2/n_2, \cdots, t_i/n_s)}$ is the volume of a sphere of radius $d$ in $\mathbf{Z}_q^n = \bigoplus_{i=1}^{s} \mathbf{Z}_q^{n_i}$ equipped with the $\theta$-metric. This is equivalent to finding all block vectors of length $n$ corresponding to the primary partition $P : n = [n_1][n_2] \cdots [n_s]$

and secondary partition $P' : t = [t_1][t_2]\cdots[t_s]$ having $\theta$-weight $d$ or less. We obtain the number of such block vectors in the following lemma:

**Lemma 3.8.** *If $V_{d,q}^{(t_i/n_1,t_2/n_2,\cdots,t_i/n_s)}$ denote the number of all block vectors in $\mathbf{Z}_q^n = \bigoplus_{i=1}^{s}\mathbf{Z}_q^{n_i}$ having $\theta$-weight equal to $d$ corresponding to the primary partition $P : n = [n_1][n_2]\cdots[n_s]$ and secondary partition $P' : t = [t_1][t_2]\cdots[t_s]$. Then $V_{d,q}^{(t_i/n_1,t_2/n_2,\cdots,t_i/n_s)}$ is given by*

$$V_{d,q}^{(t_i/n_1,t_2/n_2,\cdots,t_i/n_s)} = \sum_{(p_1,p_2,\cdots,p_s)}\left(\prod_{i=1}^{s}L(p_i,t_i,n_i)\right) \tag{3}$$

*where the summation in (3) is taken over all s-tuples $(p_1,p_2,\cdots,p_s)$ of non-negative integers satisfying*

$$\begin{aligned} 0 &\leq p_i \leq [q/2] \text{ for all } 1 \leq i \leq s; \\ 0 &\leq \sum_{i=1}^{s}p_i \leq d, \end{aligned} \tag{4}$$

and $L(p_i,t_i,n_i)$ represents the number of ways of obtaining $\theta$-weight of the $i^{th}$ ibyte equal to $p_i$ and is given by

$$L(p_i,t_i,n_i) = \sum_{j=1}^{min(p_it_i,[q/2]}\ \sum_{r_{i_0},r_{i_1}\cdots,r_{i_j}}\frac{n_i!}{r_{i0!}r_{i_1!}\cdots r_{i_j!}}e_1^{r_{i_1}}e_2^{r_{i_2}}\cdots e_j^{r_{i_j}}, \tag{5}$$

where $r_{i_k}(0 \leq k \leq j)$ are non-negative integers satisfying

$$\begin{aligned} &r_{i_0}+r_{i_1}+\cdots+r_{i_j} = n_i, r_{i_j} \geq 1, r_{i_k} \geq 0 \text{for } k \neq j; \\ &\left\lceil\frac{r_{i_1}+2r_{i_2}\cdots+jr_{i_j}}{t_i}\right\rceil = p_i. \end{aligned} \tag{6}$$

**Proof.** We consider the partition of the integer $p_it_i$, the largest entry in which is exactly equal to $j$ $(1 \leq j \leq \min(p_it_i,[q/2]))$. If $r_{i_k}(0 \leq k \leq j)$ is the number of times "$k$" or an entry equivalent to $k$ occurs then the number of ibyets of length $n_i$ having $\theta$-weight $p_i$ that can be formed by filling $n_i$ positions from the integers $0,1,2,\cdots j$ is given by

$$\frac{n_i!}{r_{i0!}r_{i_1!}\cdots r_{i_j!}}e_1^{r_{i_1}}e_2^{r_{i_2}}\cdots e_j^{r_{i_j}}. \tag{7}$$

Clearly $r_{i_0} + r_{i_1} + \cdots + r_{i_j} = n_i$, and $\left\lceil \dfrac{r_{i_1} + 2r_{i_2} \cdots + jr_{i_j}}{t_i} \right\rceil = p_i$.

Now summing (7) for all values of $r_{i_k}$ $(0 \le k \le j)$ and $j$ $(1 \le j \le \min(p_i t_i, [q/2])$ gives (5). The proof now follows from the fact that the summation in (3) is taken over all $s$-tuples $(p_1, p_2, \cdots, p_s)$ of non-negative integers where $p_i (1 \le i \le s)$ in the $\theta$-weight of the $i^{th}$ ibyte subject to constraints (4). $\square$

**Example 3.9.** Let $q = 5, n = t = 3$. Let $P = P' : 3 = [1][2]$ be a partition of $n = t = 3$. Let $d = 1$. Then $V_{1,5}^{(t_1/n_1, t_2/n_2)}$ is given by (using (3))

$$\begin{aligned} V_{1,5}^{(1/1,2/2)} &= \sum_{p_1,p_2} \left( \prod_{i=1}^{2} L(p_i, t_i, n_i) \right) \\ &= \sum_{p_1,p_2} L(p_1, 1, 1) \times L(p_2, 2, 2), \end{aligned} \tag{8}$$

where the 2-tuple $(p_1, p_2)$ of non-negative integers satisfy

$$0 \le p_1, p_2 \le 2,$$

and

$$0 \le p_1 + p_2 \le 1.$$

The feasible solutions for $(p_1, p_2)$ are

$$(p_1, p_2) = (0, 0), (1, 0) \text{ and } (0, 1).$$

We consider each of the three feasible solutions of $(p_1, p_2)$ as follows:

**Case 1.** When $(p_1, p_2) = (0, 0)$. Then

$$\begin{aligned} L(p_1, 1, 1) &= L(0, 1, 1) = 1; \\ L(p_2, 2, 2) &= L(0, 2, 2) = 1. \end{aligned}$$

**Case 2.** When $(p_1, p_2) = (0, 1)$. Then

$$\begin{aligned} L(p_1, 1, 1) &= L(1, 1, 1) = 2; \\ L(p_2, 2, 2) &= L(0, 2, 2) = 1. \end{aligned}$$

**Case 3.** When $(p_1, p_2) = (0, 1)$. Then

$$\begin{aligned} L(p_1, 1, 1) &= L(0, 1, 1) = 1; \\ L(p_2, 2, 2) &= L(1, 2, 2) = 8. \end{aligned}$$

Substituting these values in (8) gives

$$V_{1,5}^{(1/1,2/2)} = (1 \times 1) + (2 \times 1) + (1 \times 8) = 11.$$

These 11 block vectors of length $n = 3 = [1][2]$ having $\theta$-weight 1 or less over $\mathbf{Z}_5$ are given by

$$\begin{aligned} v_0 &= (0 \vdots 0\ 0), \\ v_1 &= (1 \vdots 0\ 0), \\ v_2 &= (4 \vdots 0\ 0), \\ v_3 &= (0 \vdots 1\ 0), \\ v_4 &= (0 \vdots 4\ 0), \\ v_5 &= (0 \vdots 0\ 1), \\ v_6 &= (0 \vdots 1\ 1), \\ v_7 &= (0 \vdots 4\ 1), \\ v_8 &= (0 \vdots 0\ 4), \\ v_9 &= (0 \vdots 1\ 4), \\ v_{10} &= (0 \vdots 4\ 4). \end{aligned}$$

Now we give the Hamming sphere upper bound for $\theta$-codes.

**Theorem 3.10 (Hamming Sphere packing Bound).** *Let $V$ be an $[n, k, d; P, P']$ $\theta$-code over $\mathbf{Z}_q$ corresponding to the primary-partition $P : n = [n_1][n_2] \cdots [n_s]$ and secondary-partition $P' : t = [t_1][t_2] \cdots [t_s]$. Then*

$$q^{n-k} \geq V_{[d-1/2],q}^{(t_1/n_1, t_2/n_2, \cdots, t_s/n_s)}, \tag{9}$$

*where $V_{[d-1/2],q}^{(t_1/n_1, t_2/n_2, \cdots, t_s/n_s)}$ is given by (3).*

**Proof.** The proof follows from the fact that all the $n = n_1 + n_2 + \cdots + n_s$-block vectors of $\theta$-weight $[d - 1/2]$ or less must belong to distinct cosets of the standard array and the number of available cosets in $q^{n-k}$. $\square$

**Definition 3.11.** A $\theta$-code $V$ is called a perfect code if equality holds in (9).

**Example 3.12.** Let $n = t = 2$. Let $P : P' : 2 = [1][1]$ be a partition of $n = t = 2$. Then $n_1 = t_1 = n_2 = t_2 = 1$ and $s = 2$. Let $V$ be a 5-ary $[2, 1, 3; P, P']$ $\theta$-code with parity check matrix

$$\begin{aligned} H &= (H_1 \vdots H_2) \\ &= \begin{pmatrix} 1 & \vdots & 3 \end{pmatrix}. \end{aligned}$$

The generator matrix of the code $V$ is given by

$$\begin{aligned} G &= (G_1 \vdots G_2) \\ &= \begin{pmatrix} 2 & \vdots & 1 \end{pmatrix}. \end{aligned}$$

The five codewords of the code $V$ are

$$\begin{aligned} v_0 &= (0 \vdots 0), w_\theta(v_0) = 0; \\ v_1 &= (2 \vdots 1), w_\theta(v_1) = 3; \\ v_2 &= (4 \vdots 2), w_\theta(v_2) = 3; \\ v_3 &= (1 \vdots 3), w_\theta(v_3) = 3; \\ v_4 &= (3 \vdots 4), w_\theta(v_4) = 3. \end{aligned}$$

Therefore $d_\theta = \min\limits_{\substack{x,y \in V \\ x \neq y}} d_\theta(x, y) = \min\limits_{\substack{x \in V \\ x \neq 0}} w_\theta(x) = 3$.

The equation (9) with equality in this case becomes

$$5 = V_{1,5}^{(1/1,1/1)},$$

which is true as $V_{1,5}^{(1/1,1/1)} = 5$ and can be computed similar to the computations given in Eample 3.12 as shown below:

$$V_{1,5}^{(1/1,1/1)} = (L(0,1,1) \times L(0,1,1)) + (L(1,1,1) \times L(0,1,1))$$

$$
\begin{aligned}
&+(L(0,1,1)\times L(1,1,1))\\
=&\ (1\times 1)+(2\times 1)+(1\times 2)\\
=&\ 1+2+2=5.
\end{aligned}
$$

Hence $V$ is a single $\theta$-error correcting perfect code over $\mathbf{Z}_5$.

**Theorem 3.13 (Gilbert Bound).** *Let $n,k,q,t$ be positive integers satisfying $q\geq 2, 1\leq k\leq n$ and $1\leq t\leq n$. Let $P:n=[n_1][n_2]\cdots[n_s]$, and $P':t=[t_1][t_2]\cdots[t_s]$ be the primary and secondary partitions respectively. Let $d$ be a positive integer satisfying $1\leq d\leq s[q/2]$. Then there exists an $[n,k,d;P,P']$ $\theta$-code over $\mathbf{Z}_q$ with minimum $\theta$-distance at least $d$ provided*

$$n-k\geq log_q\Big(V_{d-1,q}^{(t_1/n_1,t_2/n_2,\cdots,t_s/n_s)}\Big), \tag{10}$$

*where* $V_{d-1,q}^{(t_1/n_1,t_2/n_2,\cdots,t_s/n_s)}$ is given by (3).

**Proof.** We shall show that if (9) holds then there exists an $(n-k)\times n$ matrix $H$ over $\mathbf{Z}_q$ such that no linear combination of blocks of $H$ of $\theta$ weight $(d-1)$ or less is zero. We define an algorithm for finding the blocks $H_1,H_2,\cdots,H_s$ of $H$ where $H_i=(h_1^{(i)},h_2^{(i)},\cdots,h_{n_i}^{(i)})$ for all $1\leq i\leq s$. From the set of all $q^{n-k}$ columns vectors of length $(n-k)$ over $\mathbf{Z}_q$, we choose blocks of columns of the parity check matrix $H$ as follows:

(1) The $n_1$ column vectors in the first block $H_1$ can be any vectors chosen from the set of $q^{n-k}$ column vectors of length $n-k$ over $\mathbf{Z}_q$ satisfying

$$\lambda_1.H_1\neq 0,$$

where

$$\lambda_1=(\lambda_1^{(1)},\lambda_2^{(1)},\cdots,\lambda_{n_1}^{(1)})\in\mathbf{Z}_q^{n_1},$$

and

$$
\begin{aligned}
1\leq w_\theta(\lambda_1) &= w_\theta(\lambda_1^{(1)},\lambda_2^{(1)},\\
&\quad\cdots,\lambda_{n_1}^{(1)})\\
&\leq d-1.
\end{aligned}
$$

(2) The second block $H_2=(h_1^{(2)},h_2^{(2)},\cdots,h_{n_2}^{(2)})$ can be any set of $n_2$ column vectors of length $(n-k)$ satisfying

$$\lambda_1.H_1+\lambda_2.H_2\neq 0,$$

where for $1 \leq i \leq 2$,

$$\lambda_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{Z}_q^{n_i},$$

and

$$\begin{aligned} 1 &\leq w_\theta(\lambda_1) + w_\theta(\lambda_2) \\ &\leq d - 1. \end{aligned}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

(l) The $l^{th}$ block $H_l = (h_1^{(l)}, h_2^{(l)}, \cdots, h_{n_l}^{(l)})$ can be any set of $n_l$ column vectors of length $(n-k)$ satisfying

$$\begin{aligned} &\lambda_1.H_1 + \lambda_2.H_2 + \cdots \\ &\qquad + \lambda_l.H_l \neq 0. \end{aligned} \tag{11}$$

where

$$\begin{aligned} \lambda_i &= (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{Z}_q^{n_i} \\ &\quad \text{for all } 1 \leq i \leq l, \end{aligned}$$

and

$$\begin{aligned} 1 &\leq w_\theta(\lambda_1) + w_\theta(\lambda_2) + \cdots \\ &\quad + w_A(\lambda_l) \\ &\leq d - 1. \end{aligned} \tag{12}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

(s) The $s^{th}$ block $H_s = (h_1^{(s)}, h_2^{(s)}, \cdots, h_{n_s}^{(s)})$ can be any set of $n_s$ column vectors satisfying

$$\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_s.H_s \neq 0.$$

where

$$\begin{aligned} \lambda_i &= (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{Z}_q^{n_i} \\ &\quad \text{for all } 1 \leq i \leq s, \end{aligned}$$

and

$$\begin{aligned} 1 &\leq w_A(\theta_1) + w_A(\theta_2) + \cdots \\ &\quad + w_A(\theta_s) \\ &\leq d-1. \end{aligned}$$

If we carry out this algorithm to completion, then, $H_1, H_2, \cdots, H_s$ are the blocks of size (or length) $n_1, n_2, \cdots, n_s$ respectively of an $(n-k) \times n$ $\left(\text{where } n = \sum_{i=1}^{s} n_i\right)$ block matrix $H$ such that no linear combination of blocks of $H$ of $\theta$-weight $(d-1)$ or less is zero and this matrix is the parity check matrix for a $\theta$-code with minimum $\theta$-distance at least $d$. We show that the construction can indeed be completed. Let $l$ be an integer such that $2 \leq l \leq s$ and assume that the blocks $H_1, H_2, \cdots, H_{l-1}$ have been chosen. Then the block $H_l$ can be added to $H$ provided (11) is satisfied. The number of distinct linear combinations in (11) satisfying (12) including the pattern of all zeros is given by

$$V_{d-1,q}^{(t_1/n_1, t_2/n_2, \cdots, t_l/n_l)}$$

where $V_{d-1,q}^{(t_1/n_1, t_2/n_2, \cdots, t_l/n_l)}$ is given by (3).

As long as the set of all linear combinations occuring in (11) satisfying (12) is less than or equal to the total number of $(n-k)$-tuples, the $l^{th}$ block $H_l$ can be added to $H$. Therfore, the block $H_l$ can be added to $H$ provided that

$$q^{n-k} \geq V_{d-1,q}^{(t_1/n_1, t_2/n_2, \cdots, t_l/n_l)}$$

or

$$n-k \geq log_q\Big(V_{d-1,q}^{(t_1/n_1,t_2/n_2,\cdots,t_l/n_l)}\Big).$$

Thus the fact that the blocks $H_1, H_2, \cdots, H_s$ can be chosen follows by induction on $l$ and we get (10). $\Box$

**Corollary 3.14.** *For positive integer $t$ $\left(t \leq \dfrac{s[q/2]-1}{2}\right)$, a sufficient condition for the existence of an $[n,k,d;P,P']$ $\theta$-code $V$ over $\mathbf{Z}_q$ that correct all random block errors of $\theta$-weight $t$ or less is given by*

$$n-k \geq log_q\Big(V_{2t,q}^{(t_1/n_1,t_2/n_2,\cdots,t_s/n_s)}\Big).$$

**Proof.** The proof follows from Theorem 3.13 and the fact that to correct all errors of $\theta$-weight $t$ or less, the minimum $\theta$-weight of an ispotty byte code must be at least $2t+1$. $\Box$

**Theorem 3.15 (Varshamov Bound).** *Let $B_q(n,d;P,P')$ denotes the largest number of code vectors in an $[n,k;P,P']$ $\theta$-code over $\mathbf{Z}_q$ with $P$ : $n=[n_1][n_2]\cdots[n_s]$ and $P'$ : $t=[t_1][t_2]\cdots[t_s]$ having $\theta$-distance at least $d$. Then*

$$B_q(n,d;P,P') \geq q^{n-\lceil log_q(L)\rceil},$$

*where $L = V_{d-1,q}^{(t_1/n_1,t_2/n_2,\cdots,t_s/n_s)}$ is given by (3).*

**Proof.** By Theorem 3.13, there exists an $[n,k;P,P']$ $\theta$-code over $\mathbf{Z}_q$ with minimum $\theta$-distance at least $d$ provided

$$\begin{aligned} q^{n-k} &\geq V_{d-1,q}^{(n_1,\cdots,n_s)} = L \\ \Rightarrow n-k &\geq log_q(L) \\ \Rightarrow k &\leq n - log_q(L). \end{aligned}$$

The largest integer $k$ satisfying the above inequality is $n - \lceil log_q(L)\rceil$. Thus

$$B_q(n,d;P,P') \geq q^{n-\lceil log_q(L)\rceil}$$

where $L = V_{d-1,q}^{(t_1/n_1,t_2/n_2,\cdots,t_s/n_s)}$ is given by (3). $\Box$

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