Volume 6 -Number 1- January 2018

On the Single Particle Schrödinger Fluid

S. B. Doma[1] and A. H. El-Sharif[2]

[1] Mathematics Department, Faculty of Science, Alexandria University, Egypt

sbdoma@alexu.edu.eg

[2] Mathematics Department, Faculty of Science, Benghazi University, Libya

Abstract

The single particle Schrödinger fluid model is a fantastic concept of the application of the time-dependent Schrödinger wave equation in the collective motion of nucleons in an axially symmetric deformed nucleus. This model is formulated especially for the evaluation of moments of inertia of axially symmetric deformed nuclei. Accordingly, we apply the single-particle Schrödinger fluid model to calculate the moments of inertia of some axially-deformed nuclei in the p-shell, namely: the nuclei with mass number $8 \leq A \leq 11$.

Keywords

Second-order parabolic partial differential equations, time-dependent Schrödinger wave equation**,** single-particle Schrödinger fluid, moments of inertia.

1. Introduction

Second order parabolic partial differential equations [1] have important applications in engineering sciences, quantum mechanics and many other branches of theoretical physics. The time-dependent Schrödinger wave equation is one of the main equations of these type of differential equations. It has many different applications in nuclear, atomic, molecular and elementary-particle physics. As is well-known, its formulation depends mainly on the Hamiltonian operator which describes the motions of the quantum particles. To formulate this Hamiltonian one needs to construct certain model through which the problem can be solved. Many different models can be applied for the solutions of the different characteristics of axially-symmetric deformed nuclei. The numerical methods of solutions of these models are also of major importance in these models.

The single particle Schrödinger fluid [2-7] is one of the very interesting models which is created directly from the time-dependent Schrödinger wave equation by a suitably chosen type of complex wave functions. This model makes it possible to formulate the well-known equation of continuity, Euler's equation and Navier-Stokes equations of fluid mechanics [8] as results from the separation of the real and imaginary parts of the time-dependent Schrödinger wave equation. The single-particle potential that represents the residual interparticle interaction inside the nucleus is taken in the form of a three-dimensional anisotropic oscillator. The single particle Schrödinger fluid is a concept which is used to describe the collective motions of the nucleons in an axially symmetric deformed nucleus [9]. This concept can be applied to study the rotational motion of a deformed nucleus.

In this paper, we carry out the derivations of this model and accordingly clarify how the moment of inertia of an axially deformed nucleus can be obtained in frame-work of this model. As examples for the application of this model to the calculations of the nuclear moments of inertia we calculated the cranking-model and the rigid body-model moments of inertia of the nuclei ^{8}Li, ^{9}Li, ^{10}B and ^{11}B. The variations of the nuclear moments of inertia with respect to the deformation parameter β, which describes the deviation from the spherical case, are also given in this paper.

2. The Fluid Dynamical Equations

Let us consider a nucleus consisting of A nucleons. We assume that each nucleon in this nucleus (proton or neutron) has mass M and is moving in a single-particle potential $V\big(\boldsymbol{r}, a(t)\big)$, which is deformed with time t, through its parametric dependence on a classical shape variable $\alpha(t)$. Here, $\alpha(t)$ is assumed to be an externally prescribed function of t. Thus, the Hamiltonian for the present problem is given by [2]

$$H\big(\boldsymbol{r}, \boldsymbol{v}, a(t)\big) = -\frac{\hbar^2}{2M}\nabla^2 + V\big(\boldsymbol{r}, a(t)\big). \tag{2.1}$$

The operator ∇ in equation (2.1) appeared because in Quantum Mechanics the operator associated with the particle momentum, $\boldsymbol{p} = m\boldsymbol{v}$, is given by $-i\hbar\nabla$, where $\boldsymbol{v}$ is the particle velocity and $\hbar$ is Planck's constant divided by 2π. The single-particle time-dependent wave function $\Psi(\mathbf{r}, \alpha(t), t)$ which satisfies the time-dependent Schrödinger wave equation that describes the motion of a nucleon, is defined as

$$\mathrm{H}\big(\boldsymbol{r}, \boldsymbol{v}, \alpha(t)\big)\Psi(\boldsymbol{r}, \alpha(t), t) = i\hbar\frac{\partial}{\partial \mathrm{t}}\Psi(\boldsymbol{r}, \alpha(t), t). \tag{2.2}$$

To obtain a fluid dynamical description of the wave function $\Psi(\boldsymbol{r}, \alpha(t), t)$, we use the polar form of the wave function. We first isolate the explicit time dependence in the form

$$\Psi(\boldsymbol{r}, \alpha(t), t) = \psi\big(\boldsymbol{r}, \alpha(t)\big)\exp\left\{-\frac{i}{\hbar}\int_0^{\mathrm{t}} \epsilon\big(\alpha(t')\big)dt'\right\}, \tag{2.3}$$

where ϵ is the energy density which depends on the time through the parameter $\alpha(t)$. Then, we write the complex wave function $\psi(\boldsymbol{r}, \alpha(t))$ in the following polar form:

$$\psi\big(\boldsymbol{r}, \alpha(t)\big) = \Phi\big(\boldsymbol{r}, \alpha(t)\big)\exp\left\{-\frac{iM}{\hbar}S\big(\boldsymbol{r}, \alpha(t)\big)\right\}, \tag{2.4}$$

where $\Phi(\boldsymbol{r}, \alpha(t))$ and $S(\boldsymbol{r}, \alpha(t))$ are assumed to be real functions of $\boldsymbol{r}$ and $\alpha(t)$. Finally, we assume that the function $\Phi(\boldsymbol{r}, \alpha(t))$ is positive definite. In the case of rotation, the parameter $\alpha(t)$ becomes the angle of rotation, $\theta = \Omega t$, where Ω is the angular velocity.

Substituting equations (2.1), (2.3) and (2.4) into (2.2) we get

$$H\Psi(\boldsymbol{r},\alpha(t),t) = \exp\left\{-\frac{i}{\hbar}\int_0^t \varepsilon(\alpha(t'))dt'\right\}$$
$$\times\left\{-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r},\alpha(t))\right\}\left\{\Phi(\boldsymbol{r},\alpha(t))\exp\left\{-\frac{iM}{\hbar}S(\boldsymbol{r},\alpha(t))\right\}\right\}$$
$$= i\hbar\frac{\partial}{\partial t}\left[\Phi(\boldsymbol{r},\alpha(t))\exp\left\{-\frac{iM}{\hbar}S(\boldsymbol{r},\alpha(t))\right\}\exp\left\{-\frac{i}{\hbar}\int_0^t \varepsilon(\alpha(t'))dt'\right\}\right]$$

So that,

$$H\Psi(\boldsymbol{r},\alpha(t),t) = \exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon(\alpha(t'))dt'\right\}\exp\left\{-\frac{iM}{\hbar}S(\boldsymbol{r},\alpha(t)\right\}\times$$
$$\left[\epsilon(\alpha(t)\Phi(\boldsymbol{r},\alpha(t) + M\Phi(\boldsymbol{r},\alpha(t))\frac{\partial}{\partial t}S(\boldsymbol{r},\alpha(t)) + i\hbar\frac{\partial}{\partial t}\Phi(\boldsymbol{r},\alpha(t))\right]. \qquad (2.5)$$

Hence,

$$\left\{-\frac{\hbar^2}{2M}\nabla^2 + V(\boldsymbol{r},\alpha(t))\right\}\left\{\Phi(\boldsymbol{r},\alpha(t))\exp\left\{-\frac{iM}{\hbar}S(\boldsymbol{r},\alpha(t))\right\}\right\} =$$
$$\exp\left\{-\frac{iM}{\hbar}S(\boldsymbol{r},\alpha(t))\right\}\left[\epsilon(\alpha(t))\Phi(\boldsymbol{r},\alpha(t)) + M\Phi(\boldsymbol{r},\alpha(t))\frac{\partial}{\partial t}S(\boldsymbol{r},\alpha(t) +\right.$$
$$\left. i\hbar\frac{\partial}{\partial t}\Phi(\boldsymbol{r},\alpha(t))\right]. \qquad (2.6)$$

But we know that

$$\nabla^2\left(\Phi\exp\left\{-i\frac{MS}{\hbar}\right\}\right) = (\nabla^2\Phi)\exp\left\{-i\frac{MS}{\hbar}\right\} + \Phi\nabla^2\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right)$$
$$+2(\nabla\Phi)\cdot\nabla\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right), \qquad (2.7)$$

Also, we have

$$\nabla\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right) = -i\frac{M}{\hbar}(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\},$$

and

$$\nabla^2\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right) = -i\frac{M}{\hbar}\nabla\cdot\left[(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\}\right]$$
$$= -i\frac{M}{\hbar}\left[(\nabla^2 S)\exp\left\{-i\frac{MS}{\hbar}\right\} - i\frac{M}{\hbar}(\nabla S)\cdot(\nabla)\exp\left\{-i\frac{MS}{\hbar}\right\}\right]$$
$$= -i\frac{M}{\hbar}(\nabla^2 S)\exp\left\{-i\frac{MS}{\hbar}\right\} - \frac{M^2}{\hbar^2}(\nabla S)\cdot(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\}.$$

Substituting from the above results into equation (2.6) we get

$$i\left[\hbar\frac{\partial\Phi}{\partial t} - \frac{\hbar}{2}\Phi(\nabla^2 S) - \hbar(\nabla\Phi)\cdot(\nabla S)\right] + M\Phi\frac{\partial S}{\partial t} + \epsilon\Phi - H\Phi - \frac{M}{2}\Phi(\nabla S)\cdot(\nabla S) = 0. \qquad (2.8)$$

This equation yields, from its real and imaginary parts, a pair of coupled equations for Φ and S as follows:

$$\left[H - M\left(\frac{\partial S}{\partial t} - \frac{1}{2}\nabla S\cdot\nabla S\right)\right]\Phi = \epsilon\Phi. \tag{2.9}$$

and

$$\frac{1}{2}\Phi(\nabla^2 S) + (\nabla\Phi)\cdot(\nabla S) = \frac{\partial\Phi}{\partial t}. \tag{2.10}$$

We may call equation (2.9) modified Schrödinger equation because it differs from the usual time-independent Schrödinger equation $H\Phi = \epsilon\Phi$ by an added term which we refer to as the "dynamical modification potential"

$$V_{dyn} = -M\left[\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S)\cdot(\nabla S)\right]. \tag{2.11}$$

3. The Fluid Equations

As is well-known, the probability density of a single-particle is identified by the square of the amplitude $|\Phi|^2$ [10]. Equation (2.10), when multiplied by 2Φ, gives

$$\Phi^2\nabla^2 S + \nabla\Phi^2\cdot\nabla S = \frac{\partial\Phi^2}{\partial t}. \tag{3.1}$$

Hence, we obtain two equations the first is

$$\rho\nabla\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\rho = -\frac{\partial\rho}{\partial t}, \tag{3.2}$$

where $\boldsymbol{v}$ is the irrotational velocity and ρ is the density. This equation is the well-known equation of continuity in fluid mechanics [8]. It can be rewritten in the form:

$$\nabla\cdot(\rho\boldsymbol{v}) = -\frac{\partial\rho}{\partial t}, \tag{3.3}$$

where $\rho = \Phi^2$ and $\boldsymbol{v} = -\nabla S$.

The second equation is

$$(H + V_{dyn})\Phi = \epsilon\Phi, \tag{3.4}$$

which is a modified Schrödinger equation with

$$V_{dyn} = -M\left(\frac{\partial S}{\partial t} - \frac{1}{2}\boldsymbol{v}^2\right). \tag{3.5}$$

Equation (2.4) can be written simply as $\psi = \Phi\exp\left\{-i\frac{MS}{\hbar}\right\}$, so that

$$S = \frac{i\hbar}{2M}\ln\left(\frac{\psi}{\psi^*}\right). \tag{3.6}$$

and

$$\boldsymbol{v} = -\nabla S = \frac{i\hbar}{2M}\left[\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi}\right].$$

Therefore,

$$\boldsymbol{v} = \frac{i\hbar}{2M|\psi|^2}[\psi\nabla\psi^* - \psi^*\nabla\psi]. \tag{3.7}$$

The current of the single particle state is defined by $\boldsymbol{j} = \rho\boldsymbol{v}$ [10], so that

$$\boldsymbol{j} = \frac{i\hbar}{2M}\frac{|\Phi|^2}{|\psi|^2}[\psi\nabla\psi^* - \psi^*\nabla\psi], \tag{3.8}$$

where $\rho = |\Phi|^2$.

Since,

$$|\psi|^2 = |\Phi|^2\left|e^{-\frac{iMS}{\hbar}}\right|^2 = |\Phi|^2,$$

we finally get

$$\boldsymbol{j} = \frac{i\hbar}{2M}[\psi\nabla\psi^* - \psi^*\nabla\psi]. \tag{3.9}$$

Euler's equation for the non-viscous fluid flow is given by [8]

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho}, \tag{3.10}$$

where P is the pressure on the fluid at a point $P(\boldsymbol{r})$ at an instant of time t. For an ideal fluid, ∇P is related to the enthalpy per unit mass, w, of the fluid by the following manner

$$\frac{\nabla P}{\rho} = \nabla w. \tag{3.11}$$

Therefore, Euler's equation can be rewritten as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla w. \tag{3.12}$$

After integration and using $\boldsymbol{v} = -\nabla s$ we get

$$\frac{\partial S}{\partial t} - \frac{1}{2}v^2 = w. \tag{3.13}$$

Using also

$$v^2 = (\nabla S)^2,$$

we get

$$\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S)^2 = w, \tag{3.14}$$

where S is the velocity potential for $\boldsymbol{v}$, $(\boldsymbol{v} = -\nabla S)$ and the constant of integration in equation (3.13) is chosen here to be zero. Therefore, we can write

$$V_{dyn} = -M\left[\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S)^2\right] = -Mw, \tag{3.15}$$

and then the modified Schrödinger equation takes the form

$$(H - Mw)\Phi = \epsilon\Phi, \tag{3.16}$$

where w is now the "enthalpy" of the single-particle Schrödinger fluid [2].

Hence, we have a set of fluid dynamical equations completely analogous to those which describe a classical fluid. This set consists of the continuity equation (3.3), the Euler equation (3.12), and an equation of state (3.16). By derivations, their contents are precisely those of the original time-dependent Schrodinger equation. Hill and Wheeler [11] assumed that the single-practice Schrödinger fluid is irrotational and implicitly incompressible flow. The present formulation is specifically not restricted to incompressible flows, but allows also irrotational but compressible.

The description of the density $|\psi|^2$ as a classical fluid implies that we are assigning labels to each mass element $|\psi|^2\Delta x\Delta y\Delta z$ and considering its motion in time as described by the velocity field $\boldsymbol{v}$. However, in quantum mechanics, the quantity $|\psi|^2\Delta x\Delta y\Delta z$ is interpreted as the probability of finding the nucleon in the volume element $\Delta x\Delta y\Delta z$.

In addition to the irrotational velocity $\boldsymbol{v}$, which is a result from the fluid dynamical equation, other velocity fields which satisfy the continuity equation of the Schrödinger equation occur. Among these velocity fields are [2] the incompressible velocity field, the regular velocity field, the geometric velocity field, and the rigid body velocity field. For rotations, the rigid-body velocity field v_{rig} is defined as

$$\boldsymbol{v}_{rig} = \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{3.17}$$

It is seen that this velocity field is incompressible, regular and, also, of geometric type.

In the adiabatic approximation, where $\frac{\partial \alpha}{\partial t} \to 0$, that is the angle of rotation θ is constant of time, the collective kinetic energy of a nucleon in the nucleus is given by [2]

$$T_K = \frac{1}{2}\int \rho\boldsymbol{v}_K \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})dr. \tag{3.18}$$

and the collective kinetic energy T of the nucleus is given by [2]

$$T = \frac{1}{2}\mathrm{M}\int \rho_T\boldsymbol{v}_T \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})dr, \tag{3.19}$$

where ρ_T is the total density distribution of the nucleus and $\boldsymbol{v}_T$ is the total velocity field

$$\boldsymbol{v}_T = \frac{\sum_{occ} \rho_K \boldsymbol{v}_K}{\sum_{occ} \rho_K}. \tag{3.20}$$

4. Single Particle in the Harmonic Oscillator Potential

The single particle oscillator wave functions are taken in the form of products of three one-dimensional oscillator functions of the form [3]

$$u_{n_x} u_{n_y} u_{n_z} = u_{n_x}(\xi) u_{n_y}(\eta) u_{n_x}(\zeta), \tag{4.1}$$

where

$$u_{n_z}(\zeta) = \frac{1}{\sqrt{2^{n_z} n_z!}} \left(\frac{m\omega_z}{\pi\hbar}\right)^{\frac{1}{4}} H_{n_z}(\zeta) \exp\left(-\frac{1}{2}\zeta^2\right). \tag{4.2}$$

Similar equations hold for $u_{n_x}(\xi)$ and $u_{n_y}(\eta)$. In (4.2) $H_{n_z}(\zeta)$ is the Hermite polynomial, and the dimensionless variables are defined as [3]

$$(\xi, \eta, \zeta) = \left(\frac{\sqrt{m\omega_x}}{\hbar} x, \frac{\sqrt{m\omega_y}}{\hbar} y, \frac{\sqrt{m\omega_z}}{\hbar} z\right). \tag{4.3}$$

In the above relations, we restrict the discussion to the axially symmetric geometry for use in the case of axially-symmetric deformed nuclei. Hence, $\omega_x = \omega_y$ and the intrinsic energy of the single particle state is given by

$$E_{n_x n_y n_z} = \hbar\omega_x\left(n_x + n_y + 1\right) + \hbar\omega_z\left(n_z + \frac{1}{2}\right). \tag{4.4}$$

Using the perturbation theory, we can calculate the cranking correction to the wave function [3] explicitly and the result is

$$\mu_k = \Omega \sum_{j \neq k} \frac{|\langle j|\mathcal{L}|k\rangle|}{\epsilon_j - \epsilon_k} u_j, \tag{4.5}$$

where

$$\mu_{n_x n_y n_z} = \mu_{n_x}(\xi)\mu_{n_y n_z}(\eta,\zeta) = -\frac{\Omega u_{n_x}}{2\sqrt{\omega_y\omega_z}}\left\{\begin{matrix} \sigma\sqrt{n_y n_z}u_{n_y-1}u_{n_z-1} \\ +\frac{1}{\sigma}\sqrt{n_y(n_z+1)}u_{n_y-1}u_{n_z+1} \\ +\frac{1}{\sigma}\sqrt{n_z(n_y+1)}u_{n_y+1}u_{n_z-1} \\ +\sigma\sqrt{(n_y+1)(n_z+1)}u_{n_y+1}u_{n_z+1} \end{matrix}\right\} \tag{4.6}$$

In (4.5) and (4.6), μ_k is the first-order perturbation correction for rotation about the z-axis and the functions with subscripts n_x, n_y and n_z have arguments ξ, η and ζ, respectively. In (4.6) the quantity σ is defined by

$$\sigma = \frac{\omega_y - \omega_z}{\omega_y + \omega_z}, \tag{4.7}$$

and is a measure of the deformation of the potential.

We introduce one single parameter of deformation δ given by [12]

$$\omega_z^2 = \omega_0^2\left(1 - \frac{4}{3}\delta\right), \tag{4.8}$$

$$\omega_x^2 = \omega_y^2 = \omega_0^2\left(1 + \frac{2}{3}\delta\right). \tag{4.9}$$

The condition of constant volume of the nucleus leads to

$$\omega_x\omega_y\omega_z = const. \tag{4.10}$$

Keeping this condition in the general case together with (4.8) and (4.9), ω_0 depends on δ in the following way [12]

$$\omega_0 = \omega_0(\delta) = \omega_0^0\left\{1 - \frac{12}{9}\delta^2 - \frac{16}{27}\delta^3\right\}^{-\frac{1}{6}}, \tag{4.11}$$

where ω_0^0 is the value of $\omega_0(\delta)$ for $\delta = 0$. The quantity $\hbar\omega_0^0$ is known as the non-deformed oscillator parameter. This parameter can be calculated from the values of the total number of protons in the nucleus Z, the number of neutrons N and the mass number A as follows [3-7]

$$\hbar\omega_0^0 = \frac{38.6\mathrm{A}^{-\frac{1}{3}}}{\left[1 + \frac{1.646}{\mathrm{A}} - \frac{0.191(\mathrm{N}-\mathrm{Z})}{\mathrm{A}}\right]^2}. \tag{4.12}$$

The deformation parameter δ is related to the well-known deformation parameter β by

$$\delta = \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta. \tag{4.13}$$

The parameter β can vary in the range $-0.5 \le \beta \le 0.5$.

5. Moments of Inertia from Fluid Dynamical Viewpoint

We now examine the cranking moment of inertia in terms of the velocity fields. Bohr and Mottelson [9] showed that for harmonic oscillator case at the equilibrium deformation, where

$$\frac{d}{d\delta}\sum_{i=1}\left(E_{n_x n_y n_z}\right)_i = 0, \tag{5.1}$$

the cranking moment of inertia is identically equal to the rigid moment of inertia:

$$\mathfrak{J}_{cr} = \mathfrak{J}_{rig} = \sum_{i=1} m\langle y_i^2 + z_i^2\rangle. \tag{5.2}$$

In terms of expression (3.19) involving the velocity fields, this result asserts the equality of the collective kinetic energy of the Schrödinger fluid and that of rigidly rotating classical fluid

$$\frac{m}{2}\int \rho_T \boldsymbol{v}_T \cdot (\boldsymbol{\Omega}\times\boldsymbol{r})d\tau = \frac{1}{2}\mathfrak{J}_{rig}\Omega^2 = \frac{m}{2}\int \rho_T(\boldsymbol{\Omega}\times\boldsymbol{r})^2\, d\tau, \tag{5.3}$$

at the equilibrium deformation. We emphasize that equations (5.1) and (5.2) hold for any number of nucleons occupying any set of single particle harmonic oscillator states at the deformation defined by equilibrium condition (5.1). In particular, it holds for a one particle state. For this case, equation (5.3) becomes

$$\frac{m}{2}\int \rho_K \boldsymbol{v}_K \cdot (\boldsymbol{\Omega}\times\boldsymbol{r})d\tau = \frac{m}{2}\int \rho_K(\boldsymbol{\Omega}\times\boldsymbol{r})^2\, d\tau, \tag{5.4}$$

at the equilibrium deformation of the single particle state

$$|i\rangle \equiv |n_x n_y n_z\rangle. \tag{5.5}$$

Equation (5.4) is a remarkable identity. The scalar product of $\boldsymbol{v}_K$ and $(\boldsymbol{\Omega}\times\boldsymbol{r})$ which occurs on the left side is replaced on the right side, by the absolute square of $(\boldsymbol{\Omega}\times)$. It forces one to inquire whether the irrotational field $\boldsymbol{v}_K$ is equal to $(\boldsymbol{\Omega}\times\boldsymbol{r})$. The answer, of course, is no. For, $\boldsymbol{v}_K$ posses compressible line vortices. It could be impossible to equal the velocity field for rigid rotation $\boldsymbol{v}_{rig} = \boldsymbol{\Omega}\times\boldsymbol{r}$, which has no singularity and is everywhere incompressible and rotational. Despite this qualitative difference between $\boldsymbol{v}_K$ and the other velocity in equation (5.3), this shows that, as regards their effects under the integral upon the overall kinetic energy (or the internal parameter), these two velocity fields are equivalent at the equilibrium deformation. We note that the cranking moment of inertia $\mathfrak{J}_{cr}$ and the rigid moment of inertia $\mathfrak{J}_{rig}$ are equal only when the harmonic oscillator is at the equilibrium deformation. At other deformations, they can, and do, deviate substantially from one another [2].

The following-expressions for the cranking moment of inertia, $\mathfrak{J}_{cr}$, and the rigid-body moment of inertia, $\mathfrak{J}_{rig}$, hold [2, 3]:

$$\mathfrak{J}_{cr} = \frac{E}{\omega_0^2}\left(\frac{1}{6+2\sigma}\right)\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{3}}\left[\sigma^2(1+q) + \frac{1}{\sigma}(1-q)\right], \tag{5.6}$$

$$\Im_{rig} = \frac{E}{\omega_0^2}\left(\frac{1}{6+2\sigma}\right)\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{3}}[(1+q)+\sigma(1-q)], \tag{5.7}$$

where E is the total nuclear ground-state energy defined by

$$E = \sum_{occ}\left[\hbar\omega_x(n_x+n_y+1)+\hbar\omega_z\left(n_z+\frac{1}{2}\right)\right], \tag{5.8}$$

and q is the ratio of the summed single particle quanta in the y-and z-directions

$$q = \frac{\sum_{occ}(n_y+1)}{\sum_{occ}(n_z+1)}. \tag{5.9}$$

The quantity q is known as the anisotropy of the configuration.

6. Results and Conclusions

By assigning suitable values for the quantum numbers n_x, n_y and n_z, we constructed the ground states of the four p-shell nuclei ^{8}Li, ^{9}Li, ^{10}B and ^{11}B by filling their states with successive single-particle states as given by equation (5.8). For more details concerning this filling see Appendix-1 in ref. [7]. Accordingly, the single particle states in each nucleus are filled with the corresponding wave functions. As a result, $\Im_{cr}$ and $\Im_{rig}$ are calculated for each nucleus. Finally, the corresponding reciprocal moments $\frac{\hbar^2}{2\Im_{crank}}$ and $\frac{\hbar^2}{2\Im_{rigid}}$ are calculated.

In Figures 1, 2, 3 and 4 we present the calculated values of the reciprocal moments of inertia according to the cranking model and the rigid-body model as functions of the deformation parameter β for the nuclei ^{8}Li, ^{9}Li, ^{10}B and ^{11}B, respectively.

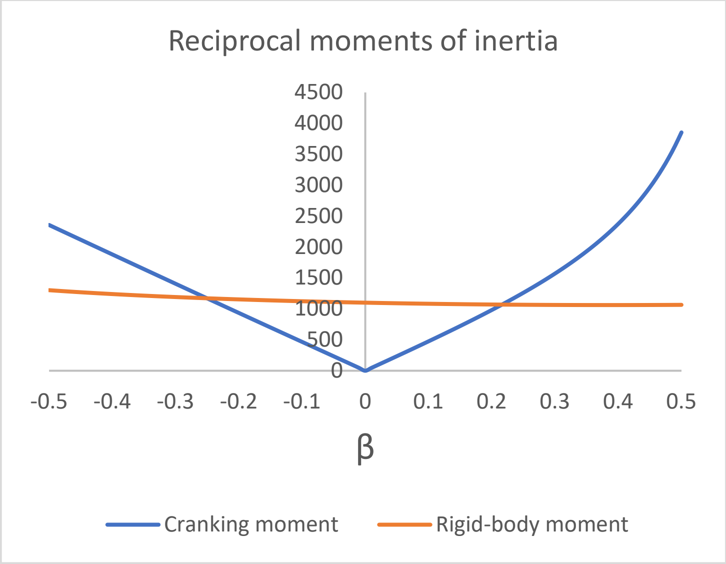

Fig.1 Reciprocal moments of inertia of the nucleus ^{8}Li

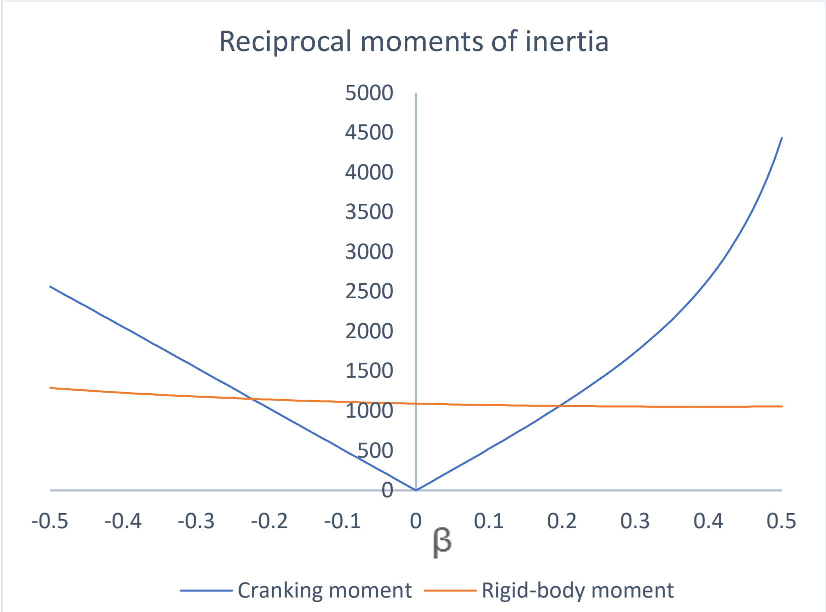

Fig.2 Reciprocal moments of inertia of the nucleus ^{9}Li

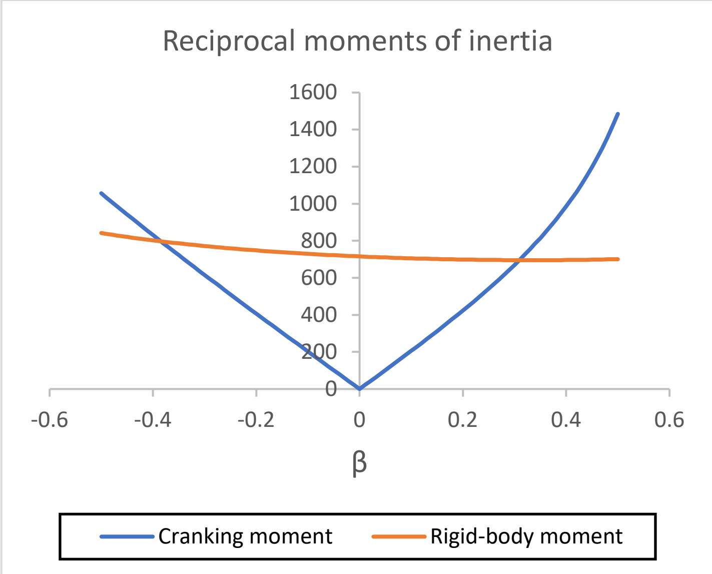

Fig. 3 Reciprocal moments of inertia of the nucleus ^{10}B

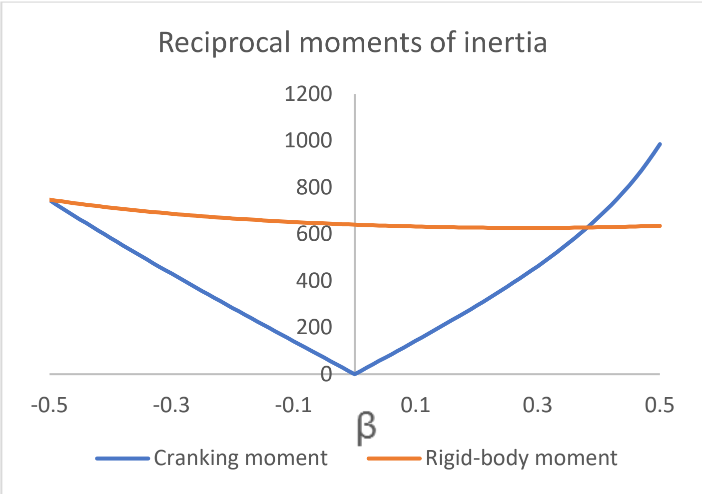

Fig. 4 Reciprocal moments of inertia of the nucleus ^{11}B

In Table-1 we present the experimental reciprocal moments of inertia of the nuclei ^{8}Li, ^{9}Li, ^{10}B and ^{11}B. The calculated values which are in good agreement with the corresponding experimental values are also given in this table. The values of the nondeformed oscillator parameter $\hbar\omega_0^0$, for the four nuclei, are given. The values of the deformation parameter β are also given in this table.

Table-1 The experimental reciprocal moments of inertia of the nuclei ^{8}Li, ^{9}Li, ^{10}B and ^{11}B. The calculated values which are in good agreement with the corresponding experimental values are given. The values of $\hbar\omega_0^0$ and β are also given in this table.

Nucleus	β	$\hbar\omega_0^0$ MeV	$\frac{\hbar^2}{2\Im_{crank}}$ KeV	$\frac{\hbar^2}{2\Im_{rigid}}$ KeV	$\frac{\hbar^2}{2\Im_{exper}}$ KeV
^{8}Li	-0.16	13.21	747.17	1140.84	750 .00
	0.16		773.01	1075.15	
^{9}Li	-0.17	14.14	872.85	1133.23	900.00
	0.17		909.16	1064.51	
^{10}B	-0.35	12.02	722.63	786.12	715.72
	0.31		697.41	695.55	
^{11}B	-0.21	12.77	298.20	668.94	297.71
	0.20		293.90	627.87	

It is seen from Figs. 1, 2, 3 and 4, and Table-1 that each nucleus has two values of the deformation parameter β which produce good agreement between the calculated and the corresponding experimental reciprocal moments of inertia for the cranking model. Concerning the rigid-body model, the calculated values are not in good agreement with the corresponding experimental values, a result which always occur with this model for most of the deformed nuclei [4-7].

Finally, we see that the single particle Schrödinger fluid has been applied successfully to the four p-shell nuclei ^{8}Li, ^{9}Li, ^{10}B and ^{11}B with a suitable choice and filling of the single-particle anisotropic harmonic oscillator states especially for the cranking-model moments. Also, the choice of the polar form for the time-dependent wave function is very accurate and produced an excellent application of the second-order parabolic partial differential equations in the calculations of the moments of inertia of deformed p-shell nuclei.

References

[1] Salaheldin Doma and Awad El-Sharif, **Second Order Partial Differential Equations and Their Applications in Physics and Engineering**, Verlag/ Publisher: Noor Publishing, Germany (2016).

[2] K. K. Kan and J. J. Griffin, Phys. Rev. C **15**: 1126 (1977).

[3] K. K. Kan and J. J. Griffin, Nucl. Phys. A **301**: 258 (1978).

[4] S. B. Doma and M. M. Amin, The single Particle Schrödinger Fluid and Moments of Inertia of the Nuclei ^{24}Mg, ^{25}Al, ^{27}Al, ^{183}W and ^{238}Pu, Intern. J. Modern Phys. E, **11**(5): 455 (2002).

[5] S. B. Doma, The Single-Particle Schrödinger Fluid and Moments of Inertia of Deformed Nuclei, Journal High Energy Phys. and Nucl. Phys., **26** (8): 836 (2002).

[6] S. B. Doma, The Single Particle Schrödinger Fluid and Moments of Inertia of Deformed Nuclei, Bulg. J. Phys. **30**:117-124 (2003).

[7] S. B. Doma and M. M. Amin, Single Particle Schrödinger Fluid and Moments of Inertia of the Even-Even Uranium Isotopes, The Open Applied Mathematics Journal, **3**: 1-6 (2009).

[8] L. D. Landau and E. M. Lifshitz, **Fluid Mechanics**, Pergamon Press, Singapore (1989).

[9] A. Bohr and B. R. Mottelson, **Nuclear Structure**, Vol. 2, Benjamin, New York (1975).

[10] J. J. Sakurai, **Modern Quantum Mechanics**, Addison-Wesley Publishing Company, Inc. (1994).

[11] D. L. Hill and J. A. Wheeler, **Nuclear Constitution and the Interpretation of Fission Phenomena**, Phys. Rev., **89** (5): 1102-1145 (1953).

[12] S. G. Nilsson, Kgl. Danske. Videnskab Selskab, mat. Fys. Medd., **29** No. 16 (1960).