On the Unitary Scheme Model of the Nucleus

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Abstract
The Hamiltonian operator of the unitary scheme model is formulated and its energy eigenvalues and eigenfunctions are given. Accordingly, the classification of the ground and excited states of a specific nucleus are given and discussed. Furthermore, the baes of this model are used to construct the bases of the nuclear supermultiplet model. Moreover, the method of evaluating the fractional parentage coefficients are also given.

Keywords: Nuclear structure, Unitary scheme model, Nuclear supermultiplet model, Fractional parentage coefficients.

1. Introduction
The nuclear shell model [1], despite a somewhat checkered career, has emerged as a useful approximation to the many-particle description of the atomic nucleus. Basically, it is perhaps the closest of all nuclear models to being unified, i.e., to describe all properties of all states of all nuclei. Unfortunately, even with the restriction to a shell structure, the number of possible states is often very large and there are few nuclei whose properties can be described without a prohibitive amount of computational labor. Thus, several sub-models of the shell model have been constructed to reduce the number of states and hence also the computational difficulty. These sub-models describe many of the physical structure of states in terms of well-defined quantum numbers.

Simultaneous with the development of the shell model has been the construction of various collective models [2]. In their simplest forms, however, these two types of models seem to have little in common: the shell model assumes of independent particle motion, whereas the collective models rely on the coherent motion of many nucleons. Neither model in its simplest form is very successful; qualitative features of nuclear energy levels can be described, but quantitative precisions is lacking. In the extreme form of the shell model, the single-particle sub-model (in which the properties of states are given by the shell model orbit of the last odd nucleon), estimates can be made for electromagnetic transition probabilities and magnetic moments (Schmidt values) which, while often of the correct order of magnitude, yet lack precision. For the collective models, on the other hand, (e.g., one based on the rotation of a nucleus deformed into a non-spherical shape) the observed approximate $J(J+1)$ dependence of bands of energy states of some nuclei can be explained and electromagnetic transitions predicted to be proportional to the square of a Clebsch-Gordan coefficient [3]. However, the moment of inertia and the other deformation characteristics must be treated as parameters [4-11].

With the development of both types of models, it soon become apparent that they may not in fact be different. The shell model goes beyond its single-particle features with the introduction of configuration mixing, while the rotational model gains some individual-particle features in the construction of the rotating intrinsic state.

The fundamental principle on which the shell model is based is that the interactions of any nucleon with all the other nucleons may be approximated by an average single-particle potential. Initially we assume this field to be spherically symmetric. In practice this field is rarely derived from any self-consistent (e.g., Hartree-Fock [12]) arguments but rather from considerations of the basic physics of the problem. Thus, since the nucleons are bound together in a finite region of space, the average field is expected to be attractive throughout the region occupied by the nucleus and to vanish everywhere else. The lowest bound states of an A-particle system are then formed by filling the lowest bound orbits in the central well obeying the Pauli principle. Since each nucleon can have spin and isospin projections of $\pm\frac{1}{2}$, each single-particle orbital state will be occupied by at most four particles: with $(m_s, m_t) = \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $\left(-\frac{1}{2}. -\frac{1}{2}\right)$

If we are concerned only with the lowest bound states, the single-particle orbits of the actual finite potential are like those of an appropriate infinite potential, e.g., the harmonic oscillator well. The advantage of using harmonic oscillator functions is that they are more amenable to mathematical manipulation. We can thus concentrate on the physical many-body aspect of the problem without additional complication in the mathematics.

There exit two basic problems in the shell model: first, the introduction of the appropriate residual interaction. Second, finding the configurations of extra core nucleons which form the eigenfunctions.

The residual interaction is supposed to take account of the effects of the nucleon-nucleon interaction which have not been included in the average central field. The effective residual interaction differs from the nuclear force between free nucleons for several reasons:

a- The presence of other nucleons inhibits the final states which may result from any interaction-quite generally this effect of the Pauli principle causes the effective residual interaction to have a longer range than the free force.

b- The nucleons polarize other nucleons, so it may be considered that the interaction is taking place between quasi particles which are not real nucleons.

Thus, one has some freedom in choosing the effective residual interaction which will lead to eigenfunctions possessing the observed features of the lowest energy levels, e.g., energies, moments, transition probabilities, etc.

The simplification technique often used in all many-body problems is to apply the invariance properties of the Hamiltonian with respect to a group G. Since the nucleus is to be considered isolated in space, it is clear that the Hamiltonian (H) will not be changed by any rotation of the coordinate system, i.e., it will be invariant with respect to the group of rotations in three dimensions, R_3. According to the representation theory of groups

[13], we can say immediately that the eigenfunctions of H can belong to a definite irreducible representation of R_3. Thus, we can label the eigenvalues by an integral or half-integral number J describing the irreducible representations. For each eigenvalue there corresponds a degeneracy of eigenfunctions corresponding at least to the dimension $(2J+1)$ of the representation. Each of these functions can be labeled with the quantum number J of the eigenvalue. In looking for groups which are invariant with respect to R_3 the one that immediately springs to mind is the space-inversion, or parity, group.

Since nucleons are fermions, they must obey the Pauli principle and hence it is possible to represent the many body eigenfunctions of the nuclear Hamiltonian in terms of the complete set of completely anti-symmetric functions. In group language the eigenfunctions belong to the totally anti-symmetric representation of the permutation group S_A, where A is the number of particles.

An approximate-symmetry group G is one for which the Hamiltonian H is approximately invariant with respect to G. This means that:

1. The matrix elements of H are negligibly small between states transforming according to different representation of G.
2. The matrix elements of H are the same between states transforming according to the same representation of G.

A broken symmetry group is one for which only the condition (1) above is true. In this case the Hamiltonian is not invariant with respect to the group G, but its eigenfunctions still transform according to a definite representation of G. Functions that belong to the same representation are, however, not degenerate. In looking for additional groups it must still be remembered that these must always allow a simultaneous classification according to R_3, i.e., states must still have a definite angular momentum.

Nuclear forces are considered largely charge independent. Thus, nuclear states will transform according to the representations of the group SU_2 of two-dimensional special unitary transformations (det =1) between the two basic isospin components $m_t = \pm\frac{1}{2}$ (i.e., the proton and the neutron). Eigenfunctions can thus be labeled with the representation label T of SU_2, and functions belonging to the same representation will be degenerate in energy. It is as well to point out that functions of the same isospin T but different projections M_T belong to nuclei of the same mass number but different charge. Thus, at this stage all nuclei of the same mass can be treated at the same time if we consider all possible isospin components.

For nuclear forces the isospin classification is treated as an approximate symmetry. In actual nuclei, however, there exists the Coulomb repulsion between protons, which of course is not charge independent. In light nuclei it is usual to treat Coulomb force as a breaking-symmetry term. Thus, states can still be classified according to isospin, but now states of the same isospin in different nuclei of the same mass number will not be degenerate in energy. Isospin then is a broken symmetry.

Another example of an approximate symmetry arises in the Wigner supermultiplet theory [13]. In this case it is assumed that nuclear forces are not only charge (isospin) independent but also largely spin independent, i.e., the dominant part of the nuclear force operates only in orbital space. Nuclear eigenfunctions can thus be considered to transform according to the representations of the group U_4 of four-dimensional unitary transformations in charge and spin space. Functions which have a definite symmetry according to U_4 also have a definite symmetry according to the group of permutations S_A between the particle numbers of charge-spin states. Remembering that the complete functions representing nuclear states must be totally anti-symmetric with respect to permutations of S_A between particle numbers in the full charge-spin-orbit space, we shall find it perhaps not hard to accept the fact that symmetry with respect to S_A in charge-spin space automatically defines the symmetry in the orbital space. The symmetry of the orbital functions is said to be adjoint to the symmetry of the charge-spin functions. All the orbital functions that transform between themselves according to a definite representation of S_A go together with the charge-spin functions of adjoint symmetry to form one totally anti-symmetric nuclear state.

Our exposition will be based on the unitary scheme model (USM) [13-39] or sometimes called the translation-invariant shell model (TISM), which is indispensable in considering the clustering effects in the p-shell nuclei. The use of oscillator functions allows us to treat freely the degrees of freedom of the cluster internal motion, but we pay for this freedom by having to be content with an incorrect asymptotic behavior of the functions used. This would require some modification of the wave-function tails at low and medium energies (of order 100- 500 MeV), but it may be acceptable at the high energies ($E_p \geq 1$GeV) and at sufficiently high energies of knocked out clusters, where the volume process dominates.

In the present paper we formulated the Hamiltonian operator of the USM and accordingly its energy eigenvalues and eigenfunctions are given. Hence, the classification of the ground and excited states of a specific nucleus are given and discussed. Furthermore, the bases of this model are used to construct the bases of the nuclear supermultiplet model. Moreover, the method of evaluating the fractional parentage coefficients are also given. Some illustrated tables for the case of $A = 7$ are given.

2. Classification of States in the USM

The USM Hamiltonian is free of spurious states. The spurious states that must be eliminated correspond to the non-zero motion of the center of mass of the whole nucleus. The unitary scheme model Hamiltonian describes the mutual motion of A nucleons in a nucleus and is of the form [18]

$$H^{(\circ)} = \sum_{i=1}^{A}\left\{\frac{1}{2m}\left(\vec{p}_i - \frac{1}{A}\sum_{k=1}^{A}\vec{p}_k\right)^2 + \frac{m\omega^2}{2}\left(\vec{r}_i - \frac{1}{A}\sum_{k=1}^{A}\vec{r}_k\right)^2\right\}, \tag{1}$$

where $\vec{r}_i$ and $\vec{p}_i$ are the coordinate and momentum operators of a quasi-particle i, m is the nucleon mass, and ω is the oscillator frequency. Let us introduce the Jacobi's transformations [13]

$$\left.\begin{aligned} x_{\alpha i} \rightleftarrows= \sum_{k=1}^{A} B_{ik\rightleftarrows}\ \xi_{\alpha k} \quad & \alpha = 1,2,3 \\ p_{\alpha i} = \sum_{k=1}^{A} B_{ik\rightleftarrows}\ \pi_{\alpha k} \quad & i,k = 1,2,\ldots,A \end{aligned}\right\}, \tag{2}$$

where the transformation matrix $\boldsymbol{B}$ satisfies the conditions

$$B_{iA} = \frac{1}{\sqrt{A}} \;, \qquad \sum_{k=1}^{A} B_{ki} = \sqrt{A}\;\delta_A^i \;,\; i = 1,2,..,A. \tag{3}$$

Appling transformations (2) to equation (1) the result is

$$H^{(0)} = \sum_{\alpha=1}^{3} \sum_{i=1}^{A-1} \left(\frac{1}{2\mathrm{m}}\; \pi_{\alpha\, i}^2 + \frac{m\omega^2}{2}\; \xi_{\alpha\, i}^2 \right). \tag{4}$$

Having the considerations of the second quantization space, we introduce the annihilation and creation oscillator quanta operators as

$$a_{\alpha\;k}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\; \xi_{\alpha\,k} - \frac{i}{\sqrt{2\mathrm{m}\hbar\omega}}\; \pi_{\alpha\,k},$$

$$a_{\alpha\;k} = \sqrt{\frac{m\omega}{2\hbar}}\; \xi_{\alpha\,k} + \frac{i}{\sqrt{2\mathrm{m}\hbar\omega}}\; \pi_{\alpha\,k} \tag{5}$$

These operators satisfy the commutation relations

$$\left[a_{\alpha\;i}\;,\; \mathrm{a}_{\beta\;\mathrm{k}}\right] = \left[a_{\alpha\;\mathrm{i}}^{+}\;,\; \mathrm{a}_{\beta\;\mathrm{k}}^{+}\right] = 0, \left[a_{\alpha\;\mathrm{i}}\;,\; \mathrm{a}_{\beta\;\mathrm{k}}^{+}\right] = \delta_{\alpha,\beta}\;\; \delta_{\mathrm{i,\,k}}\,. \tag{6}$$

The Hamiltonian operator (4) now takes the form

$$H^{(0)} = \left[\sum_{\alpha=1}^{3} \sum_{i=1}^{A-1} a_{\alpha i}^{+}\; \mathrm{a}_{\alpha i} + \frac{3}{2}\; (\mathrm{A}-1)\right]\; \hbar\omega. \tag{7}$$

It can be noticed that it is not so difficult to verify that the Hamiltonian operator (7) is invariant with respect to the transformations of the $3(A-1)$-dimensional unitary group $U_{3(A-1)}$.

The eigenfunctions of the Hamiltonian (7) are

$$\varphi_{\alpha_1\, i_1\,,\dots,\alpha_{\mathrm{N}}\; i_N} = C\;\; a_{\alpha_1\,\mathrm{i}_1}^{+}\, a_{\alpha_2\,\mathrm{i}_2}^{+} \;.\;.\;.\; \mathrm{a}_{\alpha_N\,\mathrm{i}_N}^{+}\;\; \mathrm{e}^{-\frac{m\omega}{2\,\hbar}\; \sum_{\varepsilon=1}^{3} \sum_{i=1}^{\mathrm{A}\text{-}1}\;\; \xi_{\alpha\,\mathrm{i}}^2}, \tag{8}$$

and the corresponding eigenvalues are represented by

$$E_N^{(0)} = \left[N + \frac{3}{2}\,(\mathrm{A}\text{-}1)\right]\; \hbar\omega. \tag{9}$$

Since the functions (8) are symmetric with respect to permutations of any pair of its indices, they may be used as bases for irreducible representation (IR) of a symmetric tensor of the rank N. The Young Scheme $\{N\}$ is useful for obtaining such IR. It is clear that the dimension of the representation $\{N\}$ of the group $U_{3(A-1)}$ is equal to the number of functions $\varphi_{\alpha_1\, i_1\,,\dots,\alpha_{\mathrm{N}}\; i_N}$. The bases (8) are usually denoted by

$$|A\;\Gamma\;\mathrm{M}_L; \Gamma_S\;\mathrm{M}_S\; M_T\rangle \equiv \left|A\;N\;\;\{\rho\}\;(\upsilon)\;\;\alpha\;\;[f]\;(\lambda\;\mu)\;L\;M_L; \left[\tilde{f}\right]\;\mathrm{S}\;\mathrm{T}\;\mathrm{M}_S\;\mathrm{M}_T\right\rangle, \tag{10}$$

where Γ and Γ_S are the sets of all orbital and spin-isospin quantum numbers characterizing the states, respectively. The total number of quanta N is the irreducible representation (IR) of the group $U_{3(A-1)}$. The irreducible representations of groups U_3 and $U_{(A-1)}$ are set by the same symbols $\{\rho\} = \{\rho_1, \rho_2, \rho_3\}$, where $\rho_1 \geq \rho_2 \geq \rho_3 \geq 0$ are any integers satisfying the requirements $\rho_1 + \rho_2 + \rho_3 = N$. The symbol $(\lambda\mu)$ of the SU_3 symmetry is determined by the relations $\lambda = \rho_1 - \rho_2$, $\mu = \rho_2 - \rho_3$, which enables us to find the values of the total orbital angular momentum L, by using Elliott's rule [13]. According to this rule

$L = K, K+1, \dots, K+B;$ $K = C, C-2, \dots, 1$ or 0 for $K \neq 0$,

$L = B, B-2, \dots, 1$ or 0 if $K = 0$ where $C = min(\lambda, \mu)$ and $B = max(\lambda, \mu)$.

The allowed Young Schemes [*f*] for the representation $\{\rho_1, \rho_2, \rho_3\}$ of group $U_{(A-1)}$ may be found using the formalism of plethysm, which has been described in detail in [13]. In Eqn. (10), M_L stands for the IR of the group SO_2. The representation (ν) is an IR of the group O_{A-1} and [f] is an IR of the symmetric group. S, M_S are the spin, its projection and T, M_T are the isospin, its projection which are IR of the direct product of the groups $SU_2 \times SU_2$. Among all the possible Young schemes [f], only those comprising not more than four columns should be selected. If, after that, the values S, T are to be taken for the conjugated Young diagrams $[\tilde{f}]$, we shall obtain the total list of the USM states with given quantum number *N*.

3. The Supermultiplet Model

In this section we are in prospect to study the special features of wave function. This function can be built, disregarding the internal structure of the orbital wave function of the nucleus. The supermultiplet model [13] will be based on the properties of symmetric group and irreducible tensor spaces of the unitary groups. The method of constructing the supermultiplet wave function of the nucleus is based on that simple position, which follows from the theory of symmetric group, according to

$$\alpha([A] \times [f'] \to [f]) = \delta([f'], [f]),$$
$$\alpha([1^A] \times [f'] \to [f]) = \delta([f'], [\tilde{f}]), \tag{11}$$

anti-symmetric representation $[1^A]$ is contained only in the direct product of the two conjugate representations $[f] \times [\tilde{f}]$. Therefore, the anti-symmetric wave function can be separated into orbital and spin-isospin functions by the simple binding having the following form

$$\Psi(\Gamma_\circ \Gamma_s([f][\tilde{f}])[1^A]) = \sum_{\mu\tilde{\mu}} \psi_{[f]\ \mu}(\Gamma_\circ) \rightleftarrows \psi_{[\tilde{f}] \rightleftarrows \tilde{\mu}} \rightleftarrows (\Gamma_s)\, C^{[f][\tilde{f}]\ [A]}_{\mu\ \ \tilde{\mu}\ \ [1^A]}. \tag{12}$$

In (12) $\psi_{[f]\mu}$ designates orbital and $\psi_{[\tilde{f}]\tilde{\mu}}$ spin-isospin functions characterized by the collections of orbital $\Gamma_\circ$ and spin-isospin Γ_s quantum numbers and the CGCs. of the symmetric group S_A, where

$$\Gamma_\circ = N\{\rho\}(\nu)[f]LM_L \quad \text{and } \Gamma_s = [\tilde{f}]STM_sM_T. \tag{13}$$

The totally anti-symmetric Young Scheme $[1^A] = [11 \dots 1]$ (A-times) is the irreducible representation (IR) of the group $\mathfrak{I}$. Since $[1^A] = [f] \times [\tilde{f}]$ therefore, the (IR) of the group $\mathfrak{I}$ can be reduced to direct product of two unitary groups $U_{3(A-1)}$ and U_{4^A} corresponding to the orbital and spin-isospin functions i.e.,

$$\mathfrak{I} \supset \begin{array}{c} U_{3(A-1)} \\ \times \\ U_{4^A} \end{array}. \tag{14}$$

The orbital and spin-isospin reduction-group chain is given by [13]

$$U_{3(A-1)} \supset \times \begin{matrix} SU(3) \supset SO_3 \supset SO_2 \\ U_{A-1} \supset O_{A-1} \supset S_A \end{matrix} \quad \text{and} \quad U_{4^A} \supset U_{4A} \supset \times \begin{matrix} SU_4 \supset \begin{matrix} SU_2 \\ SU_2 \\ \times \end{matrix} \\ U_A \supset S_A \end{matrix}. \tag{15}$$

Finally, the reduction chain of the group $\mathfrak{I}$ possesses the form

$$\begin{array}{ccccccccccc} & & & & & U_{A-1} & \supset & O_{A-1} & \supset & S_A & \\ & & U_{3(A-1)} & \supset & \times & & & & & & \\ & & & & & SU_3 & \supset & SO_3 & \supset & SO_2 & \\ \mathfrak{I} & \supset & \times & & & & & & & & \\ & & & & & & & SU_4 & \supset & SU_2 & \times & SU_2 \\ & & U_{4^A} & \supset & U_{4A} & \supset & & \times & & & \\ & & & & & & & U_A & \supset & S_A & \end{array} \tag{16}$$

The first coefficient in which designates spin and the second isospin of function. Let $S_A^{(S)}$ and $S_A^{(T)}$ are symmetric groups, which transpose respectively spin and isospin coordinates. Then the spin function is characterized by the diagram

$$[f_s] \equiv [\tfrac{A}{2} + S, \tfrac{A}{2} - S], \tag{17}$$

and the isospin by the diagram

$$[f_T] \equiv [\tfrac{A}{2} + T, \tfrac{A}{2} - T], \tag{18}$$

The corresponding basis is symbolically denoted by

$$\Psi = \psi_{[f_s]\rightleftarrows\rightleftarrows\mu_s}(M_s)\ \ \psi_{[f_T]\rightleftarrows\rightleftarrows\mu_T}(M_T) \tag{19}$$

which is designated through μ_S and μ_T. $[f_S]$ designates both the irreducible representation of a group S_A and irreducible representation of a group SU_2, which assigns the nucleons. According to the chain

$$S_n \supset S_{n-1} \supset \ldots \supset S_2 \supset S_1. \tag{20}$$

we have

$$[\bar{f}_s] = [\frac{A-1}{2} + \bar{S}, \frac{A-1}{2} - \bar{S}]$$

$$[\bar{\bar{f}}_s] = [\tfrac{A-2}{2} + \bar{\bar{S}}, \tfrac{A-2}{2} - \bar{\bar{S}}] \quad \text{, etc.} \tag{21}$$

Obviously, the spin-isospin function of the nucleus can be built via the binding of ideas $[f_S]$ and $[f_T]$ by the Clebsch-Gordan coefficients of the symmetric group S_A, we have

$$\Psi_{[\tilde{f}]\tilde{\mu}}(\Gamma_s) = \sum_{\mu_s\mu_T} \psi_{[f_S]\mu_s}(M_S)\, \psi_{[f_T]\mu_T}(M_T) C^{[f_S][f_T]\tilde{\alpha}[\tilde{f}]}_{\mu_S\ \mu_T\ \ \tilde{\mu}} \quad (22)$$

With the aid of formulas (12) and (22) we achieved the construction of the supermultiplet wave function of the nucleus, and for this purpose it is sufficiently enough to use only two types of Clebsch-Gordan coefficients of the group S_A satisfying the relations

$$[f] \times [\tilde{f}] \to [1^A], \quad (23)$$

and

$$[f_S] \times [f_T] \to \tilde{\alpha}[\tilde{f}] \quad (24)$$

Subsequently, the supermultiplet wave function of the nucleus is designated by

$$\Psi = \Psi(\Gamma_\circ([f]([f_S][f_T])\tilde{\alpha}[\tilde{f}])[1^A]M_S M_T) \quad (25)$$

Let us further consider that the states of nuclei must be described by the quantum number J of the total angular momentum, $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Hence, it follows that the collection $\Gamma_\circ$ must include the quantum number of the total orbital angular momentum L and of its projection M_L therefore, if we replaced $\Gamma_\circ$ in (25) with new collection $\Gamma_\circ L M_L$ and to connect the momenta L and S in J, then the supermultiplet wave function of the nucleus with the most complete characteristic takes the following form:

$$\Psi(\Gamma_\circ \pi \ \rightleftarrows L \ \rightleftarrows ([f]([f_S][f_T])\ \tilde{\alpha}\ [\tilde{f}])[1^A]\, J M_J M_T) \quad (26)$$

The quantum number π, determining the parity of the orbital wave function of the nucleus, in (24), is explicitly extracted in addition. We find $[\tilde{f}]$, together with the representation of a group S_A, also does designate irreducible representation of a group SU_4, and this IR is given in the chain

$$SU_4 \supset\rightleftarrows\rightleftarrows\ SU_2 \times \mathrm{SU}_2,$$

which in turn does lead to quantum numbers S, M_S and T, M_T. This sense acquires and bringing $[f_S] \rightleftarrows\times [f_T] \to \tilde{\alpha}\,[\tilde{f}]$, which with the use of transformation properties of group SU_4 should be written in the form:

$$[\tilde{f}] = \sum_{[f_S]\,[f_T]} \tilde{\alpha}\left([\tilde{f}] \supset [f_S]\,[f_T]\right) [f_S]\,[f_T]., \quad (27)$$

here $\tilde{\alpha}$ is the number of repetitions of identical $[f_S]$, $[f_T]$ in $[\tilde{f}]$; it can be designated in abbreviated form as follows:

$$[\tilde{f}] = \sum_{S\rightleftarrows\rightleftarrows T} \tilde{\alpha}\left([\tilde{f}] \supset \mathrm{S\,T}\right)[\mathrm{S, T}]. \quad (28)$$

All the possible values of S and T, of the diagram $[\tilde{f}]$, are obtained in the brackets $[S, T]$. Instead of the IR $[\ \tilde{f}_1 \rightleftarrows \tilde{f}_2\tilde{f}_3\tilde{f}_4]$ of the group U_4 we use the simpler IR $[\tilde{f}_1 - \tilde{f}_4\ , \tilde{f}_2 - \tilde{f}_4, \tilde{f}_3 - \tilde{f}_4,\ 0]$ of the unitary unimodular subgroup SU_4. It also follows from this circumstance of the supermultiplet structures (i.e. the possible collections of S and T) that the orbital diagrams of the form of $[4 \dots 4f^0]$ and $[f^0]$ are identical. Therefore, for example, in this sense instead of the diagram $[4 \dots 4321]$ it is enough to examine the diagram of $[321]$, etc. Moreover, the diagrams $[f_1\ f_2\ \dots]$ and $[\dots\ 4-f_2,\ 4-f_1]$ also have identical supermultiplet structure. The less symmetrical diagrams of Young contain more high values of spin and isospin of the nucleus.

For example, in the case of the nuclei with $A = 7$, we have IR of the symmetric group S_7 which we deal with in our calculations, denoted $by\,[f] = [43], [421], [331]$ and $[322]$ which have $[\tilde{f}]$ = $[2^3 1]$, $[3211]$, $[41^3]$, $[322]$, and $[331]$ respectively. By using equations (18) and (19) we can obtain the values of S and T, as follows:
First, we write all inner product of $A = 7$ characterized by $[f_1\ f_2]$ such that $f_1 + f_2 = A$ as follows:

$$[61] \times [52] = [61] + [52] + [511] + [43] + [421]$$

$$[61] \times [43] = [52] + [43] + [421] + [331]$$

$$[52] \times [52] = [7] + [61] + 2[52] + [511] + [43] + 2[421] + [413] + [331] + [322]$$

$$[52] \times [43] = [61] + [52] + [511] + [43] + 2[421] + [331] + [322] + [3211]$$

$$[43] \times [43] = [7] + [61] + [52] + [511] + [43] + [421] + [413] + [331] + [322] + + [3211] + [2^3 1].$$

Second, we search in what inner product there exist $[\tilde{f}]$, we find $[2^3 1]$ in the inner product $[43] \times [43]$; $[3211]$ in $[52] \times [43]$ and $[43] \times [43]$; $[413]$ in $[52] \times [52]$ and $[43] \times [43]$; $[322]$ in $[52] \times [52], [52] \times [43]$ and $[43] \times [43]$; and finally $[331]$ in $[61] \times [43]$, $[52] \times [52]$ and $[43] \times [43]$.

Third, we compare the diagrams of $[f_S]$ and $[f_T]$ and the corresponding for each $[\tilde{f}]$ we find for $[2^3 1]$ that (S,T) = $(\frac{1}{2}, \frac{1}{2})$; for $[3211]$ we find (S,T) = $(\frac{3}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2})$; for $[41^3]$ we find (S,T) = $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2})$; for $[322]$ we find (S,T) = $(\frac{3}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2})$; finally for $[331]$ we find (S,T) = $(\frac{5}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{5}{2})$, $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. By this way for any $[f]$ we can find (S,T).

In Table-1 we show the classification of the USM-bases for the ground state of ^{7}Li with $N = 3, 5, 7$; $(J^\pi, T) = (\frac{3}{2}^-, \frac{1}{2})$.

Table-1 Classification of the USM-bases for the ground state of ^{7}Li. $N = 3, 5, 7$; $(J^\pi, T) = (\frac{3}{2}^-, \frac{1}{2})$.

No.	N	$\{\rho\}$	(υ)	$[f]$	α	L	S	T
1	3	{3}	(3)	[43]		1	1/2	1/2
2	3	{21}	(21)	[421]		1	1/2	1/2
3	3	{21}	(21)	[421]		1	3/2	1/2
4	3	{21}	(21)	[421]		3	3/2	1/2
5	5	{5}	(3)	[43]		1	1/2	1/2
6	5	{5}	(5)	[43]	2	1	1/2	1/2
7	5	{5}	(5)	[421]	2	1	1/2	1/2
8	5	{5}	(5)	[421]	2	1	3/2	1/2
9	5	{5}	(5)	[421]	2	3	3/2	1/2
10	5	{5}	(5)	[331]		1	1/2	1/2
11	5	{5}	(5)	[331]		1	3/2	1/2
12	5	{5}	(5)	[331]		3	3/2	1/2
13	5	{41}	(3)	[43]		1	1/2	1/2
14	5	{41}	(3)	[43]		2	1/2	1/2
15	5	{41}	(21)	[421]		1	1/2	1/2
16	5	{41}	(21)	[421]		1	3/2	1/2
17	5	{41}	(21)	[421]		2	1/2	1/2
18	5	{41}	(21)	[421]		2	3/2	1/2
19	5	{41}	(21)	[421]		3	3/2	1/2
20	5	{41}	(41)	[43]	2	1	1/2	1/2
21	5	{41}	(41)	[421]	4	1	1/2	1/2
22	5	{41}	(41)	[421]	4	1	3/2	1/2
23	5	{41}	(41)	[421]	4	2	1/2	1/2
24	5	{41}	(41)	[421]	4	2	3/2	1/2
25	5	{41}	(41)	[421]	4	3	3/2	1/2
26	5	{41}	(41)	[331]		1	1/2	1/2
27	5	{41}	(41)	[331]		1	3/2	1/2
28	5	{41}	(41)	[331]		2	1/2	1/2
29	5	{41}	(41)	[331]		2	3/2	1/2
30	5	{41}	(41)	[331]		3	3/2	1/2
31	5	{41}	(41)	[322]		1	1/2	1/2
32	5	{41}	(41)	[322]		1	3/2	1/2
33	5	{41}	(41)	[322]		2	1/2	1/2
34	5	{41}	(41)	[322]		2	3/2	1/2
35	5	{41}	(41)	[322]		3	3/2	1/2
36	5	{32}	(3)	[43]		1	1/2	1/2
37	5	{32}	(3)	[43]		2	1/2	1/2
38	5	{32}	(21)	[421]		1	1/2	1/2
39	5	{32}	(21)	[421]		1	3/2	1/2
40	5	{32}	(21)	[421]		2	1/2	1/2
41	5	{32}	(21)	[421]		2	3/2	1/2
42	5	{32}	(21)	[421]		3	3/2	1/2
43	5	{32}	(32)	[43]	2	1	1/2	1/2
44	5	{32}	(32)	[43]	2	2	1/2	1/2
45	5	{32}	(32)	[421]	3	1	1/2	1/2

46	5	{32}	(32)	[421]	3	1	3/2	1/2
47	5	{32}	(32)	[421]	3	2	1/2	1/2
48	5	{32}	(32)	[421]	3	2	3/2	1/2
49	5	{32}	(32)	[421]	3	3	3/2	1/2
50	5	{32}	(32)	[331]	2	1	1/2	1/2
51	5	{32}	(32)	[331]	2	1	3/2	1/2
52	5	{32}	(32)	[331]	2	2	1/2	1/2
53	5	{32}	(32)	[331]	2	2	3/2	1/2
54	5	{32}	(32)	[331]	2	3	3/2	1/2
55	5	{32}	(32)	[322]		1	1/2	1/2
56	5	{32}	(32)	[322]		1	3/2	1/2
57	5	{32}	(32)	[322]		2	1/2	1/2
58	5	{32}	(32)	[322]		2	3/2	1/2
59	5	{32}	(32)	[322]		3	3/2	1/2
60	5	{311}	(21)	[421]		0	3/2	1/2
61	5	{311}	(21)	[421]		2	1/2	1/2
62	5	{311}	(21)	[421]		2	3/2	1/2
63	5	{311}	(311)	[421]	2	0	3/2	1/2
64	5	{311}	(311)	[421]	2	2	1/2	1/2
65	5	{311}	(311)	[421]	2	2	3/2	1/2
66	5	{311}	(311)	[331]		0	3/2	1/2
67	5	{311}	(311)	[311]		2	1/2	1/2
68	5	{311}	(311)	[311]		2	3/2	1/2
69	5	{311}	(311)	[322]		0	3/2	1/2
70	5	{311}	(311)	[322]		2	1/2	1/2
71	5	{311}	(311)	[322]		2	3/2	1/2
72	5	{221}	(21)	[421]		1	1/2	1/2
73	5	{221}	(21)	[421]		1	3/2	1/2
74	5	{221}	(221)	[43]		1	1/2	1/2
75	5	{221}	(221)	[421]		1	1/2	1/2
76	5	{221}	(221)	[421]		1	3/2	1/2
77	5	{221}	(221)	[331]		1	1/2	1/2
78	5	{221}	(221)	[331]		1	3/2	1/2
79	5	{221}	(221)	[322]		1	1/2	1/2
80	5	{221}	(221)	[322]		1	3/2	1/2
81	7	{7}	(3)	[43]		1	1/2	1/2
82	7	{7}	(5)	[43]	2	1	1/2	1/2
83	7	{7}	(5)	[421]	2	1	1/2	1/2
84	7	{7}	(5)	[421]	2	1	3/2	1/2
85	7	{7}	(5)	[421]	2	3	3/2	1/2
86	7	{7}	(5)	[331]		1	1/2	1/2
87	7	{7}	(5)	[331]		1	3/2	1/2
88	7	{7}	(5)	[331]		3	3/2	1/2
89	7	{7}	(7)	[43]	4	1	1/2	1/2
90	7	{7}	(7)	[421]	5	1	1/2	1/2
91	7	{7}	(7)	[421]	5	1	3/2	1/2
92	7	{7}	(7)	[421]	5	3	3/2	1/2
93	7	{7}	(7)	[331]	2	1	1/2	1/2
94	7	{7}	(7)	[331]	2	1	3/2	1/2
95	7	{7}	(7)	[331]	2	3	3/2	1/2

96	7	{7}	(7)	[322]		1	1/2	1/2
97	7	{7}	(7)	[322]		1	3/2	1/2
98	7	{7}	(7)	[322]		3	3/2	1/2
99	7	{61}	(3)	[43]		1	1/2	1/2
100	7	{61}	(3)	[43]		2	1/2	1/2
101	7	{61}	(21)	[421]		1	1/2	1/2
102	7	{61}	(21)	[421]		1	3/2	1/2
103	7	{61}	(21)	[421]		2	1/2	1/2
104	7	{61}	(21)	[421]		2	3/2	1/2
105	7	{61}	(21)	[421]		3	3/2	1/2
106	7	{61}	(5)	[43]	2	1	1/2	1/2
107	7	{61}	(5)	[43]	2	2	1/2	1/2
108	7	{61}	(5)	[421]	2	1	1/2	1/2
109	7	{61}	(5)	[421]	2	1	3/2	1/2
110	7	{61}	(5)	[421]	2	2	1/2	1/2
111	7	{61}	(5)	[421]	2	2	3/2	1/2
112	7	{61}	(5)	[421]	2	3	3/2	1/2
113	7	{61}	(5)	[331]		1	1/2	1/2
114	7	{61}	(5)	[331]		1	3/2	1/2
115	7	{61}	(5)	[331]		2	1/2	1/2
116	7	{61}	(5)	[331]		2	3/2	1/2
117	7	{61}	(5)	[331]		3	3/2	1/2
118	7	{61}	(41)	[43]	2	1	1/2	1/2
119	7	{61}	(41)	[43]	2	2	1/2	1/2
120	7	{61}	(41)	[421]	4	1	1/2	1/2
121	7	{61}	(41)	[421]	4	1	3/2	1/2
122	7	{61}	(41)	[421]	4	2	1/2	1/2
123	7	{61}	(41)	[421]	4	2	3/2	1/2
124	7	{61}	(41)	[421]	4	3	3/2	1/2
125	7	{61}	(41)	[331]		1	1/2	1/2
126	7	{61}	(41)	[331]		1	3/2	1/2
127	7	{61}	(41)	[331]		2	1/2	1/2
128	7	{61}	(41)	[331]		2	3/2	1/2
129	7	{61}	(41)	[331]		3	3/2	1/2
130	7	{61}	(41)	[322]		1	1/2	1/2
131	7	{61}	(41)	[322]		1	3/2	1/2
132	7	{61}	(41)	[322]		2	1/2	1/2
133	7	{61}	(41)	[322]		2	3/2	1/2
134	7	{61}	(41)	[322]		3	3/2	1/2
135	7	{61}	(61)	[43]	6	1	1/2	1/2
136	7	{61}	(61)	[43]	6	2	1/2	1/2
137	7	{61}	(61)	[421]	12	1	1/2	1/2
138	7	{61}	(61)	[421]	12	1	3/2	1/2
139	7	{61}	(61)	[421]	12	2	1/2	1/2
140	7	{61}	(61)	[421]	12	2	3/2	1/2
141	7	{61}	(61)	[421]	12	3	3/2	1/2
142	7	{61}	(61)	[331]	5	1	1/2	1/2
143	7	{61}	(61)	[331]	5	1	3/2	1/2
144	7	{61}	(61)	[331]	5	2	1/2	1/2
145	7	{61}	(61)	[331]	5	2	3/2	1/2

146	7	{61}	(61)	[331]	5	3	3/2	1/2
147	7	{61}	(61)	[322]	4	1	1/2	1/2
148	7	{61}	(61)	[322]	4	1	3/2	1/2
149	7	{61}	(61)	[322]	4	2	1/2	1/2
150	7	{61}	(61)	[322]	4	2	3/2	1/2
151	7	{61}	(61)	[322]	4	3	3/2	1/2
152	7	{52}	(3)	[43]	2	2	1/2	1/2
153	7	{52}	(21)	[421]		2	1/2	1/2
154	7	{52}	(21)	[421]		2	3/2	1/2
155	7	{52}	(21)	[421]		3	3/2	1/2
156	7	{52}	(5)	[43]	2	2	1/2	1/2
157	7	{52}	(5)	[421]	2	2	1/2	1/2
158	7	{52}	(5)	[421]	2	2	3/2	1/2
159	7	{52}	(5)	[421]	2	3	3/2	1/2
160	7	{52}	(5)	[331]		2	1/2	1/2
161	7	{52}	(5)	[331]		2	3/2	1/2
162	7	{52}	(5)	[331]		3	3/2	1/2
163	7	{52}	(41)	[43]	2	2	1/2	1/2
164	7	{52}	(41)	[421]	4	2	1/2	1/2
165	7	{52}	(41)	[421]	4	2	3/2	1/2
166	7	{52}	(41)	[421]	4	3	3/2	1/2
167	7	{52}	(41)	[331]		2	1/2	1/2
168	7	{52}	(41)	[331]		2	3/2	1/2
169	7	{52}	(41)	[331]		3	3/2	1/2
170	7	{52}	(41)	[322]		2	1/2	1/2
171	7	{52}	(41)	[322]		2	3/2	1/2
172	7	{52}	(41)	[322]		3	3/2	1/2
173	7	{52}	(32)	[43]	2	2	1/2	1/2
174	7	{52}	(32)	[421]	3	2	1/2	1/2
175	7	{52}	(32)	[421]	3	2	3/2	1/2
176	7	{52}	(32)	[421]	3	3	3/2	1/2
177	7	{52}	(32)	[331]	2	2	1/2	1/2
178	7	{52}	(32)	[331]	2	2	3/2	1/2
179	7	{52}	(32)	[331]	2	3	3/2	1/2
180	7	{52}	(32)	[322]		2	1/2	1/2
181	7	{52}	(32)	[322]		2	3/2	1/2
182	7	{52}	(32)	[322]		3	3/2	1/2
183	7	{52}	(52)	[43]	7	2	1/2	1/2
184	7	{52}	(52)	[421]	13	2	1/2	1/2
185	7	{52}	(52)	[421]	13	2	3/2	1/2
186	7	{52}	(52)	[421]	13	3	3/2	1/2
187	7	{52}	(52)	[331]	7	2	1/2	1/2
188	7	{52}	(52)	[331]	7	2	3/2	1/2
189	7	{52}	(52)	[331]	7	3	3/2	1/2
190	7	{52}	(52)	[322]	6	2	1/2	1/2
191	7	{52}	(52)	[322]	6	2	3/2	1/2
192	7	{52}	(52)	[322]	6	3	3/2	1/2
193	7	{43}	(3)	[43]		1	1/2	1/2
194	7	{43}	(3)	[43]		2	1/2	1/2
195	7	{43}	(21)	[421]		1	1/2	1/2

196	7	{43}	(21)	[421]		1	3/2	1/2
197	7	{43}	(21)	[421]		2	1/2	1/2
198	7	{43}	(21)	[421]		2	3/2	1/2
199	7	{43}	(21)	[421]		3	3/2	1/2
200	7	{43}	(41)	[43]	2	1	1/2	1/2
201	7	{43}	(41)	[43]	2	2	1/2	1/2
202	7	{43}	(41)	[421]	4	1	1/2	1/2
203	7	{43}	(41)	[421]	4	1	3/2	1/2
204	7	{43}	(41)	[421]	4	2	1/2	1/2
205	7	{43}	(41)	[421]	4	2	3/2	1/2
206	7	{43}	(41)	[421]	4	3	3/2	1/2
207	7	{43}	(41)	[331]		1	1/2	1/2
208	7	{43}	(41)	[331]		1	3/2	1/2
209	7	{43}	(41)	[331]		2	1/2	1/2
210	7	{43}	(41)	[331]		2	3/2	1/2
211	7	{43}	(41)	[331]		3	3/2	1/2
212	7	{43}	(41)	[322]		1	1/2	1/2
213	7	{43}	(41)	[322]		1	3/2	1/2
214	7	{43}	(41)	[322]		2	1/2	1/2
215	7	{43}	(41)	[322]		2	3/2	1/2
216	7	{43}	(41)	[322]		3	3/2	1/2
217	7	{43}	(32)	[43]	2	1	1/2	1/2
218	7	{43}	(32)	[43]	2	2	1/2	1/2
219	7	{43}	(32)	[421]	3	1	1/2	1/2
220	7	{43}	(32)	[421]	3	1	3/2	1/2
221	7	{43}	(32)	[421]	3	2	1/2	1/2
222	7	{43}	(32)	[421]	3	2	3/2	1/2
223	7	{43}	(32)	[421]	3	3	3/2	1/2
224	7	{43}	(32)	[331]	2	1	1/2	1/2
225	7	{43}	(32)	[331]	2	1	3/2	1/2
226	7	{43}	(32)	[331]	2	2	1/2	1/2
227	7	{43}	(32)	[331]	2	2	3/2	1/2
228	7	{43}	(32)	[331]	2	3	3/2	1/2
229	7	{43}	(32)	[322]		1	1/2	1/2
230	7	{43}	(32)	[322]		1	3/2	1/2
231	7	{43}	(32)	[322]		2	1/2	1/2
232	7	{43}	(32)	[322]		2	3/2	1/2
233	7	{43}	(32)	[322]		3	3/2	1/2
234	7	{43}	(43)	[43]	4	1	1/2	1/2
235	7	{43}	(43)	[43]		2	1/2	1/2
236	7	{511}	(41)	[322]		0	3/2	1/2
237	7	{511}	(41)	[322]		2	1/2	1/2
238	7	{511}	(41)	[322]		2	3/2	1/2
239	7	{511}	(311)	[421]	2	0	3/2	1/2
240	7	{511}	(311)	[421]	2	2	1/2	1/2
241	7	{511}	(311)	[421]	2	2	3/2	1/2
242	7	{511}	(311)	[331]		0	3/2	1/2
243	7	{511}	(311)	[331]		2	1/2	1/2
244	7	{511}	(311)	[331]		2	3/2	1/2
245	7	{511}	(311)	[322]		0	3/2	1/2

246	7	{511}	(311)	[322]		2	1/2	1/2
247	7	{511}	(311)	[322]		2	3/2	1/2
248	7	{511}	(511)	[43]	2	2	1/2	1/2
249	7	{511}	(511)	[421]	8	0	3/2	1/2
250	7	{511}	(511)	[421]	8	2	1/2	1/2
251	7	{511}	(511)	[421]	8	2	3/2	1/2
252	7	{511}	(511)	[331]	4	0	3/2	1/2
253	7	{511}	(511)	[331]	4	2	1/2	1/2
254	7	{511}	(511)	[331]	4	2	3/2	1/2
255	7	{511}	(511)	[322]	4	0	3/2	1/2
256	7	{511}	(511)	[322]	4	2	1/2	1/2
257	7	{511}	(511)	[322]	4	2	3/2	1/2
258	7	{421}	(3)	[43]		1	1/2	1/2
259	7	{421}	(3)	[43]		2	1/2	1/2
260	7	{421}	(21)	[421]	2	1	1/2	1/2
261	7	{421}	(21)	[421]	2	1	3/2	1/2
262	7	{421}	(21)	[421]	2	2	1/2	1/2
263	7	{421}	(21)	[421]	2	2	3/2	1/2
264	7	{421}	(21)	[421]	2	3	3/2	1/2
265	7	{421}	(41)	[43]	2	1	1/2	1/2
266	7	{421}	(41)	[43]	2	2	1/2	1/2
267	7	{421}	(41)	[421]	4	1	1/2	1/2
268	7	{421}	(41)	[421]	4	1	3/2	1/2
269	7	{421}	(41)	[421]	4	2	1/2	1/2
270	7	{421}	(41)	[421]	4	2	3/2	1/2
271	7	{421}	(41)	[421]	4	3	3/2	1/2
272	7	{421}	(41)	[331]		1	1/2	1/2
273	7	{421}	(41)	[331]		1	3/2	1/2
274	7	{421}	(41)	[331]		2	1/2	1/2
275	7	{421}	(41)	[331]		2	3/2	1/2
276	7	{421}	(41)	[331]		3	3/2	1/2
277	7	{421}	(41)	[322]		1	1/2	1/2
278	7	{421}	(41)	[322]		1	3/2	1/2
279	7	{421}	(41)	[322]		2	1/2	1/2
280	7	{421}	(41)	[322]		2	3/2	1/2
281	7	{421}	(41)	[322]		3	3/2	1/2
282	7	{421}	(32)	[43]	2	1	1/2	1/2
283	7	{421}	(32)	[43]	2	2	1/2	1/2
284	7	{421}	(32)	[421]	3	1	1/2	1/2
285	7	{421}	(32)	[421]	3	1	3/2	1/2
286	7	{421}	(32)	[421]	3	2	1/2	1/2
287	7	{421}	(32)	[421]	3	2	3/2	1/2
288	7	{421}	(32)	[421]	3	3	3/2	1/2
289	7	{421}	(32)	[331]	2	1	1/2	1/2
290	7	{421}	(32)	[331]	2	1	3/2	1/2
291	7	{421}	(32)	[331]	2	2	1/2	1/2
292	7	{421}	(32)	[331]	2	2	3/2	1/2
293	7	{421}	(32)	[331]	2	3	3/2	1/2
294	7	{421}	(32)	[322]		1	1/2	1/2
295	7	{421}	(32)	[322]		1	3/2	1/2

296	7	{421}	(32)	[322]		2	1/2	1/2
297	7	{421}	(32)	[322]		2	3/2	1/2
298	7	{421}	(32)	[322]		3	3/2	1/2
299	7	{421}	(311)	[421]	2	1	1/2	1/2
300	7	{421}	(311)	[421]	2	1	3/2	1/2
301	7	{421}	(311)	[421]	2	2	1/2	1/2
302	7	{421}	(311)	[421]	2	2	3/2	1/2
303	7	{421}	(311)	[421]	2	3	3/2	1/2
304	7	{421}	(311)	[331]		1	1/2	1/2
305	7	{421}	(311)	[331]		1	3/2	1/2
306	7	{421}	(311)	[331]		2	1/2	1/2
307	7	{421}	(311)	[331]		2	3/2	1/2
308	7	{421}	(311)	[331]		3	3/2	1/2
309	7	{421}	(311)	[322]		1	1/2	1/2
310	7	{421}	(311)	[322]		1	3/2	1/2
311	7	{421}	(311)	[322]		2	1/2	1/2
312	7	{421}	(311)	[322]		2	3/2	1/2
313	7	{421}	(311)	[322]		3	3/2	1/2
314	7	{421}	(221)	[43]		1	1/2	1/2
315	7	{421}	(221)	[43]		2	1/2	1/2
316	7	{421}	(221)	[421]		1	1/2	1/2
317	7	{421}	(221)	[421]		1	3/2	1/2
318	7	{421}	(221)	[421]		2	1/2	1/2
319	7	{421}	(221)	[421]		2	3/2	1/2
320	7	{421}	(221)	[421]		3	3/2	1/2
321	7	{421}	(221)	[331]		1	1/2	1/2
322	7	{421}	(221)	[331]		1	3/2	1/2
323	7	{421}	(221)	[331]		2	1/2	1/2
324	7	{421}	(221)	[331]		2	3/2	1/2
325	7	{421}	(221)	[331]		3	3/2	1/2
326	7	{421}	(221)	[322]		1	1/2	1/2
327	7	{421}	(221)	[322]		1	3/2	1/2
328	7	{421}	(221)	[322]		2	1/2	1/2
329	7	{421}	(221)	[322]		2	3/2	1/2
330	7	{421}	(221)	[322]		3	3/2	1/2
331	7	{421}	(421)	[43]	4	1	1/2	1/2
332	7	{421}	(421)	[43]	4	2	1/2	1/2
333	7	{421}	(421)	[421]	9	1	1/2	1/2
334	7	{421}	(421)	[421]	9	1	3/2	1/2
335	7	{421}	(421)	[421]	9	2	1/2	1/2
336	7	{421}	(421)	[421]	9	2	3/2	1/2
337	7	{421}	(421)	[421]	9	3	3/2	1/2
338	7	{421}	(421)	[331]	6	1	1/2	1/2
339	7	{421}	(421)	[331]	6	1	3/2	1/2
340	7	{421}	(421)	[331]	6	2	1/2	1/2
341	7	{421}	(421)	[331]	6	2	3/2	1/2
342	7	{421}	(421)	[331]	6	3	3/2	1/2
343	7	{421}	(421)	[322]	6	1	1/2	1/2
344	7	{421}	(421)	[322]	6	1	3/2	1/2
345	7	{421}	(421)	[322]	6	2	1/2	1/2

346	7	{421}	(421)	[322]	6	2	3/2	1/2
347	7	{421}	(421)	[322]	6	3	3/2	1/2
348	7	{331}	(21)	[421]		1	1/2	1/2
349	7	{331}	(21)	[421]		1	3/2	1/2
350	7	{331}	(32)	[43]	2	1	1/2	1/2
351	7	{331}	(32)	[421]	3	1	1/2	1/2
352	7	{331}	(32)	[421]	3	1	3/2	1/2
353	7	{331}	(32)	[331]	2	1	1/2	1/2
354	7	{331}	(32)	[331]	2	1	3/2	1/2
355	7	{331}	(32)	[322]		1	1/2	1/2
356	7	{331}	(32)	[322]		1	3/2	1/2
357	7	{331}	(311)	[421]	2	1	1/2	1/2
358	7	{331}	(311)	[421]	2	1	3/2	1/2
359	7	{331}	(311)	[331]		1	1/2	1/2
360	7	{331}	(311)	[331]		1	3/2	1/2
361	7	{331}	(311)	[322]		1	1/2	1/2
362	7	{331}	(311)	[322]		1	3/2	1/2
363	7	{331}	(331)	[43]		1	1/2	1/2
364	7	{331}	(331)	[421]	4	1	1/2	1/2
365	7	{331}	(331)	[421]	4	1	3/2	1/2
366	7	{331}	(331)	[331]	2	1	1/2	1/2
367	7	{331}	(331)	[331]	2	1	3/2	1/2
368	7	{331}	(331)	[322]	2	1	1/2	1/2
369	7	{331}	(331)	[322]	2	1	3/2	1/2
370	7	{322}	(3)	[43]		1	1/2	1/2
371	7	{322}	(21)	[421]		1	1/2	1/2
372	7	{322}	(21)	[421]		1	3/2	1/2
373	7	{322}	(32)	[43]	2	1	1/2	1/2
374	7	{322}	(32)	[421]	3	1	1/2	1/2
375	7	{322}	(32)	[421]	3	1	3/2	1/2
376	7	{322}	(32)	[331]	2	1	1/2	1/2
377	7	{322}	(32)	[331]	2	1	3/2	1/2
378	7	{322}	(32)	[322]		1	1/2	1/2
379	7	{322}	(32)	[322]		1	3/2	1/2
380	7	{322}	(221)	[43]		1	1/2	1/2
381	7	{322}	(221)	[421]		1	1/2	1/2
382	7	{322}	(221)	[421]		1	3/2	1/2
383	7	{322}	(221)	[331]		1	1/2	1/2
384	7	{322}	(221)	[331]		1	3/2	1/2
385	7	{322}	(221)	[322]		1	1/2	1/2
386	7	{322}	(221)	[322]		1	3/2	1/2
387	7	{322}	(322)	[43]		1	1/2	1/2
388	7	{322}	(322)	[421]	2	1	1/2	1/2
389	7	{322}	(322)	[421]	2	1	3/2	1/2
390	7	{322}	(322)	[331]	2	1	1/2	1/2
391	7	{322}	(322)	[331]	2	1	3/2	1/2
392	7	{322}	(322)	[322]	2	1	1/2	1/2
393	7	{322}	(322)	[322]	2	1	3/2	1/2
394	7	{43}	(43)	[421]	8	1	1/2	1/2
395	7	{43}	(43)	[421]	8	1	3/2	1/2

396	7	{43}	(43)	[421]	8	2	1/2	1/2
397	7	{43}	(43)	[421]	8	2	3/2	1/2
398	7	{43}	(43)	[421]	8	3	3/2	1/2
399	7	{43}	(43)	[331]	4	1	1/2	1/2
400	7	{43}	(43)	[331]	4	1	3/2	1/2
401	7	{43}	(43)	[331]	4	2	1/2	1/2
402	7	{43}	(43)	[331]	4	2	3/2	1/2
403	7	{43}	(43)	[331]	4	3	3/2	1/2
404	7	{43}	(43)	[322]	3	1	1/2	1/2
405	7	{43}	(43)	[322]	3	1	3/2	1/2
406	7	{43}	(43)	[322]	3	2	1/2	½
407	7	{43}	(43)	[322]	3	2	3/2	1/2
408	7	{43}	(43)	[322]	3	3	3/2	1/2
409	7	{511}	(21)	[421]		0	3/2	1/2
410	7	{511}	(21)	[421]		2	1/2	1/2
411	7	{511}	(21)	[421]		2	3/2	1/2
412	7	{511}	(41)	[43]	2	2	1/2	1/2
413	7	{511}	(41)	[421]	4	0	3/2	1/2
414	7	{511}	(41)	[421]	4	2	1/2	1/2
415	7	{511}	(41)	[421]	4	2	3/2	1/2
416	7	{511}	(41)	[331]	4	0	3/2	1/2
417	7	{511}	(41)	[331]	4	2	1/2	1/2
418	7	{511}	(41)	[331]	4	2	3/2	½

4. The Fractional Parentage Decomposition in the USM

Doma and Machabeli [33] modified the recurrence method, introduced by Vanagas [13], for calculating the two-particle orbital FPCs. of nuclei with any mass number A and they applied the new method for selected bases of the USM corresponding to nuclei with $A = 6$ and $N = 2,$ and 4. Also, Doma and Youssef [19], and Doma [20] constructed bases of the USM for nuclei with $A = 3$ and $0 \leq N \leq 10$ and calculated the two- particle orbital FPCs. for these bases. Furthermore, Doma [21,23] constructed the bases of the USM for nuclei with $A = 4, N = 0, 2, 4, 6, 8$ and nuclei with $A = 6$ and $N = 2, 4,$ and 6. Also, Doma and Gharib [40] constructed the bases of the USM for the nucleus ^{5}He corresponding to number of quanta of excitation $N = 1$ and 3 and calculated the two-particle orbital FPCs. for these nuclei.

Moreover, a general and direct method for calculating the two-particle spin-isospin FPCs. has been introduced by Doma [17,18,34] and then applied to the calculation of some coefficients of the two-particle spin-isospin FPCs. for nuclei with $A = 4$ and 6.

The resulting two-particle FPCs. are very useful in calculating the matrix elements of any kind of the two-particle operators, such as the central, the tensor, the spin orbit and the quadratic spin orbit operators [23,29,36,37,41]. In the following we are interested in the case where the quantum number Γ of the supermultiplet function Ψ assumes the following classification.

$$\Gamma = [f]\ \Gamma_0\, \Gamma_S, \tag{29}$$

where $[f]$ is an irreducible representation (IR) of the symmetric group of A objects, S_A, Γ_0 and Γ_S are the sets of all the other orbital and spin-isospin quantum numbers, respectively.

Following the method introduced by Vanagas [13], the expansion coefficient of the decomposition of the nuclear wave function with A particles into its two subfunctions corresponding to A' and A'' particles, $B_{\Gamma'\,\Gamma_0\,\Gamma''\,,\Gamma}$, can be factorized in terms of product of orbital and spin-isospin parts as follows:

$$B_{[f']\,\Gamma_0'\,\Gamma_S'\,[f'']\,\Gamma_0''\,\Gamma_S'',\,[f]\,\Gamma_0\,\Gamma_S} = \sum_{\alpha_0\,\tilde{\alpha}_0} A^{[f']\quad[f'']}_{\Gamma_0'\,\Gamma_0'',\,[f]\,\alpha_0\,\Gamma_0} \times C^{[\tilde{f}']\,[\tilde{f}'']\,\tilde{\alpha}_0\,[\tilde{f}]}_{\Gamma_S'\quad\Gamma_S''\qquad\Gamma_S}\; C^{[f]\qquad\qquad[\tilde{f}]}_{\alpha_0\,[f']\,[f'']\quad\tilde{\alpha}_0\,[\tilde{f}']\,[\tilde{f}'']}\,, \tag{30}$$

here α_0 and $\tilde{\alpha}_0$ are repetition indices in the orbital and the spin-isospin states, respectively and the last factor under the sum is the isoscaler factor of the C.G.C. of the symmetric group S_A. The second factor in the right-hand side of Eqn. (30) is a CGC of the unitary group in four dimensions U_4. The first factor in Eqn. (30) is the orbital FPC. This factor can be factorized as follows:

$$A^{[f']\quad[f'']}_{\Gamma_0'\,\Gamma_0'',\,[f]\,\alpha_0\,\Gamma_0} = \delta_{\rho'+\rho'',\,\rho}\;\sum_\beta A^{(\{\rho\}\;[f']\;[f''])}_{\{\rho'\}\,(\upsilon')\,\{\rho''\}\,(\upsilon'')\,\beta\,,\,(\upsilon)\,[f]\,\alpha_0} \times C^{\{\rho'\}\qquad\{\rho''\}\qquad\beta\;\{\rho\}}_{\gamma'\;L'\,M_L''\;\;\gamma''\,L''\,M_L''\;\;\gamma\,L\,M_L} \tag{31}$$

where $\{\rho'\}$, $\{\rho''\}$ and $\{\rho\}$are IR of the unitary group in $3(A'-1)$, $3(A''-1)$ and $3(A-1)$ dimensions, respectively. They are also IR of the group SU_3, simultaneously. L', L'' and L are orbital-angular momentum quantum numbers of the sets A', A'' and A particles, respectively. M'_L, M''_L and M_L are the z-projections of L', L'' and L, β shows how many times $\{\rho\}$ appears in the multiplication $\{\rho'\}\times\{\rho''\}$. Similarly, the repetition index γ shows how many times L appears in $\{\rho\}$, and for γ' and γ''. Concerning the repetitions index α_0 of Eqn. (30) it shows how many times the IR [f] appears in (ν), where (ν) is an IR of the orthogonal group O_{A-1}. The last factor in (31) represents the CGC of the group SU_3. Substituting from Eqn. (31) into Eqn. (30) we get:

$$B_{[f']\,\Gamma_0'\,\Gamma_S'\,[f'']\,\Gamma_0''\,\Gamma_S'',\,[f]\,\Gamma_0\,\Gamma_S} = \sum_{\alpha_0\,\tilde{\alpha}_0} \delta_{\rho'+\rho'',\,\rho}\;\sum_\beta A^{(\{\rho\}\;[f']\;[f''])}_{\{\rho'\}\,(\upsilon')\,\{\rho''\}\,(\upsilon'')\,\beta\,,\,(\upsilon)\,[f]\,\alpha_0} \times C^{\{\rho'\}\qquad\{\rho''\}\qquad\beta\;\{\rho\}}_{\gamma'\;L'\,M_L''\;\;\gamma''\,L''\,M_L''\;\;\gamma\,L\,M_L}\; C^{[\tilde{f}']\,[\tilde{f}'']\,\tilde{\alpha}_0\,[\tilde{f}]}_{\Gamma_S'\quad\Gamma_S''\qquad\Gamma_S}\; C^{[f]\qquad\qquad[\tilde{f}]}_{\alpha_0\,[f']\,[f'']\quad\tilde{\alpha}_0\,[\tilde{f}']\,[\tilde{f}'']} \tag{32}$$

Eqn. (32) gives the total many-particle FPC of the supermultiplet of the nuclear wave function.

For $A''=2$ and $A'=A-2$, the two-particle total FPC has the form:

$$B_{\overline{[f]}\;\overline{\Gamma_0}\;\overline{\Gamma_S}\,,\,[f_{12}]\,\varepsilon\,\ell\,m_\ell\,s\,m_s\,t\,m_t\,;\,[f]\,\Gamma_0\,\Gamma_S} = A^{(\{\rho\}\quad\overline{[f]}\quad[f_{12}])}_{\{\overline{\rho}\}\,(\upsilon)\,\{\varepsilon\}\,(\upsilon_{12})\,;\,(\upsilon)\,[f]}\; C^{\{\overline{\rho}\}\;\;\{\varepsilon\}\;\;\{\rho\}}_{\overline{L}\,\overline{M}_L\;\ell\,m_\ell\;L\,M_L}$$

$$\times\ C^{[\bar{f}]}_{S\ M_S \bar{T}\ M_T}\ {}^{[\tilde{f}_{12}]}_{s\ m_s\ t\ m_t}\ {}^{[\tilde{f}]}_{S\ M_S\ T\ M_T}\ \sqrt{\frac{d_{[\bar{f}]}}{d_{[f]}}} \tag{33}$$

where $[f_{12}] = [2]$ or $[11]$, $\Gamma_0 \equiv N\,\{\rho\}\,(\nu)\,L\,ML, \Gamma s \equiv \left[\tilde{f}\right] S\,MS\,T\,MT$, $d_{[\bar{f}]}$and $d_{[f]}$are the dimensions of the IR $[\bar{f}]$ and $[f]$, respectively. The first coefficients in the right-hand side of Eqn. (33) which are usually called the two-particle orbital FPC are calculated as follows:

$$A^{(\{\rho\}\quad [\bar{f}]\quad [f_{12}])}_{\{\bar{\rho}\}\ (v)\ \{\varepsilon\}\ (v_{12})\ ;\ (v)\ [f]} \equiv \sum_{\bar{\rho}'} \sum_{i=1,2} \acute{A}^{(\{\rho\}\quad [\bar{f}])}_{\{\bar{\rho}'\}\ \{\bar{\rho}\}\ (\bar{v}),\ (v)\ [f]\ [\bar{f}_i]}$$

$$\times\ U^{([f]\ \overline{[f]}\ )}_{\overline{[f_i]}\ \ [f_{12}]}\ D^{\{\rho\}}_{\{\bar{\rho}\}\ ,\ \{\bar{\rho}'\}}(a'), \tag{34}$$

where

$$D^{\rho}_{\bar{\rho}\ ,\ \bar{\rho}'}(a') = D^{j}_{m'\ m}(a')\ \left\langle (\bar{\rho}\ (\varepsilon_2\ \varepsilon_1)\ \rho_{12})\ \rho \,\middle|\, \left((\bar{\rho}\ \varepsilon_2)\ \bar{\rho}'\ \varepsilon_1\right)\ \rho \right\rangle$$

$$\times\ \left\langle \left((\bar{\rho}\ \varepsilon_s)\ \bar{\rho}\ \varepsilon_a\right) \bar{\rho} \,\middle|\, (\bar{\rho}\ (\varepsilon_s\ \varepsilon_a)\ \rho_{12})\ \rho \right\rangle. \tag{35}$$

Here,

$$\varepsilon_1 = \rho - \bar{\rho}',\ \varepsilon_2 = \bar{\rho}' - \bar{\rho}\ ,\ \varepsilon_s = \bar{\rho} - \bar{\rho}\ ,\ \varepsilon_a = \rho - \bar{\rho}\ ,\ \rho_{12} = \{\rho_1, \rho_2\} \subset \varepsilon_2 \times \varepsilon_1\ ,$$

$$j = \frac{1}{2}(\rho_1 - \rho_2)\ ,\ m' = \frac{1}{2}(\varepsilon_2 - \varepsilon_1)\ ,\ m = \frac{1}{2}(\varepsilon_2 - \varepsilon_1) \text{ and}$$

$$a' = \begin{vmatrix} a_1' & a_2' \\ -a_2' & a_1' \end{vmatrix}\ ,\ a_1' = \sqrt{\frac{A}{2\,(A-1)}}\ \ ,\ a_2' = -\sqrt{\frac{A-2}{2(A-1)}}$$

In Eqn. (35) $D^{j}_{m'\ m}(a')$ are the matrix elements of the IR of the group SU_2 and are given by

$$D^{j}_{m'\ m} = \sum_i \frac{[(j+m')!\ (j-m')!\ (j+m)!\ (j-m)!]^{\frac{1}{2}}}{i!\ (j-m'-i)!\ (j+m-i)!\ (i-m+m')!}$$

$$\times\ a_1'^{\,j+m-i}\ \ a_2'^{\,i}\ \ a_3'^{\,i-m+m'}\ \ a_4'^{\,j-m'-i} \tag{36}$$

The last two factors in the right-hand side of Eqn. (35) are the recoupling matrix elements of the IR of the SU_3 group and are given by Vanagas [13]. The factors $U^{([f]\ \overline{[f]}\ )}_{\overline{[f_i]}\ \ [f_{12}]}$ in (34) are the recoupling matrix elements of the IR of the group S_A.

The coefficients $\acute{A}^{(\{\rho\}\quad [\bar{f}])}_{\{\bar{\rho}'\}\ \{\bar{\rho}\}\ (\bar{v}),\ (v)\ [f]\ [\bar{f}_i]}$, of eqn. (34) are orthonormal solutions of the following system of linear homogeneous equations

$$\sum_{\overline{\rho}'} \left\langle \{\rho\}\ \overline{\{\rho\}}\ \overline{\{\rho\}} \right| \mathrm{P}_{\mathrm{A-1,A}} \left| \{\rho\}\ \overline{\{\rho\}}'\ \overline{\{\rho\}} \right\rangle \grave{\mathrm{A}}_{\overline{\{\rho\}}'\ \overline{\{\rho\}}\ \overline{(\upsilon)},(\upsilon)\ [f]\ \overline{[f]}'}^{\{\rho\}\ \overline{[f]}}$$

$$= \sum_{\overline{f}} D_{\overline{[f]}\ \overline{[f]}\ ,\ \overline{[f]}'\overline{[f]}}^{[f]} \left(P_{\mathrm{A-1,\,A}}\right) \grave{\mathrm{A}}_{\overline{\{\rho\}}\ \overline{\{\rho\}}\ \overline{(\upsilon)},(\upsilon)\ [f]\ \overline{[f]}}^{\{\rho\}\ \overline{[f]}}, \tag{37}$$

where the matrix elements $\left\langle \{\rho\}\ \overline{\{\rho\}}\ \overline{\{\rho\}} \right| \mathrm{P}_{\mathrm{A-1,A}} \left| \{\rho\}\ \overline{\{\rho\}}'\ \overline{\{\rho\}} \right\rangle$ can be obtained from $D^{\rho}_{\overline{\rho}\ ,\ \overline{\rho}'}(a')$ of Eqn. (35), by replacing the determinant a' by a, where

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{vmatrix}\ ,\ \mathrm{a}_1 = \frac{1}{A-1} \ \text{ and } \ \mathrm{a}_2 = \frac{\sqrt{A(A-2)}}{A-1}$$

The second factor in the right-hand side of Eqn. (32) is the CGC of the SU_3 group and can be calculated from the chain of groups $SU_3 \supset R_3$ where R_3 is the rotational group in 3-dimensions, as follows:

$$C^{\overline{\{\rho\}}\ \ \{\varepsilon\}\ \ \{\rho\}}_{\overline{L}\ \overline{M}_L\ \ell\ \mathrm{m}_\ell\ \mathrm{L}\ \mathrm{M}_L} = \left(\overline{L}\ \overline{M}_L\ ,\ \ell\ \mathrm{m}_\ell\ |\mathrm{L}\ \mathrm{M}_L\right)\ \mathrm{C}^{\overline{\{\rho\}}\ \ \{\varepsilon\}\ \ \{\rho\}}_{\overline{L}\ \ \ \ell\ \ \ \mathrm{L}}, \tag{38}$$

where $\left(\overline{L}\ \overline{M}_L\ ,\ \ell\ \mathrm{m}_\ell\ |\mathrm{L}\ \mathrm{M}_L\right)$ is a CGC of the group R_3 and $C^{\overline{\{\rho\}}\ \ \{\varepsilon\}\ \ \{\rho\}}_{\overline{L}\ \ \ \ell\ \ \ \mathrm{L}}$ is an isoscalar of the SU_3 group. The isoscalar factor of Eqn. (38) can be rewritten in the form

$$C^{\overline{\{\rho\}}\ \ \{\varepsilon\}\ \ \{\rho\}}_{\overline{L}\ \ \ \ell\ \ \ \mathrm{L}} = \left\langle \left(\overline{\lambda}\ \overline{\mu}\right) \overline{L}, \left(\lambda_a\, \mu_a\right) \ell \middle| (\lambda\, \mu)\ \mathrm{L} \right\rangle, \tag{39}$$

$where\ \lambda_a = \varepsilon_1 - \varepsilon_2\ ,\ \mu_a = \varepsilon_2\ ,\overline{\lambda} = \overline{\rho}_1 - \overline{\rho}_2\ ,\ \overline{\mu} = \overline{\rho}_2 - \overline{\rho}_3\ ,\lambda = \rho_1 - \rho_2$ and $\mu = \rho_2 - \rho_3$.

All the isoscalar factors needed in our calculations can be found in refs. [13,18,34]

Tables-2

Orbital FPC

2.1 N=3, $[\bar{f}] = [41], [f_{12}] = [2]$

$\{\rho\}(\upsilon)[f]$ \ $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	{3}(3)[43]	{21}(21)[421]
{3}{3}(1)	0.25	0
{3}{3}(3)	0.25	0
{3}{2}(2)	0.8660	0
{3}{1}(1)	0.25	0
{1}{1}(1)	0.25	0.7071
{21}{1}(1)	0	0.7071

2.2 N=3, $[\bar{f}] = [41], [f_{12}] = [11]$

$\{\rho\}(\upsilon)[f]$ \ $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	{21}(21)[421]
{2}{2}(2)	-.9984
{2}{1}(1)	-.0488
{11}{1}(1)	0.0282

2.3 N=3, $[\bar{f}] = [32], [f_{12}] = [2]$

$\{\rho\}(\upsilon)[f]$ \ $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	{3}(3)[43]	{21}{21}[421]
{3}{3}(3)	**1**	0
{21}{21}(21)	0	1

2.4 N=3, $[\bar{f}] = [32], [f_{12}] = [11]$

$\{\rho\}(\upsilon)[f]$ \ $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	{3}(3)[43]	{21}{21}[421]
{2}{2}(2)	**-1**	-1

2.5 N=3, $[\bar{f}] = [311]$

$(\upsilon)[f][f_{12}]$ \ $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(21)[421][2]	(21)[421][11]
{21}{21}(21)	1	0
{11}{11}(11)	0	-1

2.6 $\{\rho\} = \{21\}, [\overline{f}] = [221], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(21)[421]
{21}{21}(21)	1

2.7 $\{\rho\} = \{5\}, [\overline{f}] = [41], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(3)[43]	(5)[421]
$\{5\}_1\{5\}(5)$	0.0404	0.0455
$\{5\}_2\{5\}(5)$	0.0404	-.0455
{5}{5}(3)	0.0286	0.0114
{5}{5}(1)	0.0286	-.0114
$\{5\}_1\{4\}(4)$	0.3780	0.0000
$\{5\}_2\{4\}(4)$	0.3780	0.0000
{5}{3}(3)	0.2000	0.0979
{3}{3}(3)	0.2000	-.0979
{5}{3}(1)	0.0369	0.0651
{3}{3}(1)	-.0369	0.0651
{5}{2}(2)	0.4583	0.0680
{3}{2}(2)	0.5669	-.0550
{5}{1}(1)	0.2353	-.3465
{3}{1}(1)	0.0148	0.8487
{1}{1}(1)	-.1991	-.3465

2.8 $\{\rho\} = \{5\}, [\overline{f}] = [41], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]
$\{4\}_1\{4\}(4)$	-.5963
$\{4\}_2\{4\}(4)$	-.5963
{4}{3}(3)	-.4728
{4}{2}(2)	-.0422
{2}{2}(2)	0.0066
{4}{1}(1)	0.1783
{2}{1}(1)	0.1783

2.9 $\{\rho\} = \{5\}, [\bar{f}] = [32], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(3)[43]	(5)[421]	(5)[331]
$\{5\}_1\{5\}(5)$	0.4082	0.0945	0.0408
$\{5\}_2\{5\}(5)$	0.4082	-.0945	0.0408
{5}{5}(3)	0.4082	0.0000	-.0816
{5}{3}(3)	-.5000	0.0000	0.1000
{3}{3}(3)	0.5000	0.0000	0.1000
{5}{4}(4)	0.0000	0.3660	-.5523
{5}{4}(2)	0.0000	0.3660	0.5523
{5}{2}(2)	0.0000	0.7319	-.3000
{3}{2}(2)	0.0000	0.4226	0.5196

2.10 $\{\rho\} = \{5\}, [\bar{f}] = [32], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(3)[43]	(5)[421]	(5)[331]
{4}{4}(4)	-.7071	0.0000	-.6601
{4}{4}(2)	-.7071	0.0000	0.6601
{4}{3}(3)	0.0000	0.5774	0.0000
{4}{2}(2)	0.0000	-.4082	0.3105
{2}{2}(2)	0.0000	-.7071	-.1793

2.11 $\{\rho\} = \{5\}, [\bar{f}] = [311], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]	(5)[331]
$\{5\}_1\{5\}(5)$	0.5652	0.1890
$\{5\}_2\{5\}(5)$	0.5652	0.1890
{5}{5}(3)	0.5652	-.3780
{5}{3}(3)	-.1443	0.5000
{3}{3}(3)	0.1443	0.5000
{5}{4}(4)	0.0000	0.5345

2.12 $\{\rho\} = \{5\}, [\bar{f}] = [311], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]
{4}{4}(4)	0.5204
{4}{3}(3)	0.8539

2.13 $\{\rho\} = \{5\}, [\bar{f}] = [221], [f_{12}] = [2]$

$(v)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{v})$	(5)[421]
{5}{5}(5)	0.7559
{5}{4}(4)	0.6547

2.14 $\{\rho\} = \{5\}, [\bar{f}] = [221], [f_{12}] = [11]$

$(v)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{v})$	(5)[331]
{4}{4}(4)	1

2.15 $\{\rho\} = \{7\}, [\bar{f}] = [41], [f_{12}] = [2]$

$(v)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{v})$	(3)[43]	(5)[421]	$(v)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{v})$	(3)[43]	(5)[421]
{7}{1}(1)	0.1103	-.1977	{5}{4}(2)	0.2673	0.0000
{5}{1}(1)	-.0274	-.0510	$\{7\}_1\{4\}(4)$	0.2160	0.0000
{3}{1}(1)	-.0274	0.0510	$\{5\}_1\{4\}(4)$	0.2673	0.0000
{1}{1}(1)	0.1103	0.1977	$\{7\}_2\{4\}(4)$	0.2160	0.0000
{7}{2}(2)	0.2186	0.0000	$\{5\}_2\{4\}(4)$	0.2673	0.0000
{5}{2}(2)	-.0290	-.4515	{7}{5}(1)	0.2539	0.0000
{3}{2}(2)	-.0205	0.6386	{5}{5}(1)	0.2539	0.0000
{7}{3}(1)	-.1000	0.2761	{7}{5}(3)	0.2539	0.0000
{5}{3}(1)	0.0863	0.0000	{5}{5}(3)	0.2539	0.0000
{3}{3}(1)	-.1000	-.2761	$\{7\}_1\{5\}(5)$	0.2539	0.0000
{7}{3}(3)	-.1000	0.2761	$\{5\}_1\{5\}(5)$	0.2539	0.0000
{5}{3}(3)	0.0863	0.0000	$\{7\}_2\{5\}(5)$	0.2539	0.0000
{3}{3}(3)	-.1000	-.2761	$\{5\}_2\{5\}(5)$	0.2539	0.0000
{7}{4}(2)	0.2160	0.0000			

2.16 $\{\rho\} = \{7\}, [\bar{f}] = [41], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\rho\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]
{6}{1}(1)	0.5512
{4}{1}(1)	0.6039
{2}{1}(1)	0.5512
{6}{2}(2)	0.0321
{4}{2}(2)	-.1361
{2}{2}(2)	0.0717
{6}{3}(1)	0.0269
{4}{3}(1)	0.0269
{6}{3}(3)	0.0269
{4}{3}(3)	0.0269

2.17 $\{\rho\} = \{7\}, [\bar{f}] = [32], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\rho\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(3)[43]	(5)[421]	(5)[331]	(5)[322]
{7}{4}(4)	0.4373	-.4484	-.5365	-.1897
{5}{4}(4)	0.5411	0.3624	0.1858	-.5477
{7}{4}(2)	0.4373	-.4484	0.5365	0.1897
{5}{4}(2)	0.5411	0.3624	-.1858	0.5477
{7}{5}(3)	0.0730	0.0303	0.1601	-.2864
{5}{5}(3)	-.0730	0.0303	0.1601	-.2864
$\{7\}_1\{5\}(5)$	0.0730	0.0303	0.1601	0.2864
$\{5\}_1\{5\}(5)$	-.0730	0.0303	0.1601	0.2864
$\{7\}_2\{5\}(5)$	0.0730	0.0303	-.3203	0.0000
$\{5\}_2\{5\}(5)$	-.0730	0.0303	-.3203	0.0000
{7}{2}(2)	0.0000	0.4270	0.0000	0.0000
{5}{2}(2)	0.0000	0.2700	0.0000	0.0000
{3}{2}(2)	0.0000	0.1909	0.0000	0.0000
{5}{3}(3)	0.0000	0.1952	0.0000	0.0000
{7}{3}(3)	0.0000	0.0000	0.1543	0.0000
{3}{3}(3)	0.0000	0.0000	-.1543	0.0000

2.18 $\{\rho\} = \{7\}, [\bar{\bar{f}}] = [32], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\rho\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(3)[43]	(5)[421]	(5)[331]
{6}{4}(4)	0.5175	-.2501	0.1718
{4}{4}(4)	0.2134	0.6064	-.2315
{6}{4}(2)	0.5175	-.2501	-.1718
{4}{4}(2)	0.2134	0.6064	0.2315
{6}{5}(3)	-.3528	0.0000	0.0000
$\{6\}_1\{5\}(5)$	-.3528	0.0000	0.0000
$\{6\}_2\{5\}(5)$	-.3528	0.0000	0.0000
{6}{2}(2)	0.0000	-.1318	0.0000
{4}{2}(2)	0.0000	-.1863	0.0000
{2}{2}(2)	0.0000	-.2946	0.0000
{6}{3}(3)	0.0000	-.0165	-.6457
{4}{3}(3)	0.0000	0.0165	-.6457

2.19 $\{\rho\} = \{7\}, [\bar{f}] = [311], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]	(5)[331]
$\{7\}_1\{5\}(5)$	0.4629	0.0000
$\{5\}_1\{5\}(5)$	0.4629	0.0000
$\{7\}_2\{5\}(5)$	-.1548	0.0452
$\{5\}_2\{5\}(5)$	0.1548	0.0452
{7}{4}(4)	0.3553	0.1964
{5}{4}(4)	-.1231	0.5669
{7}{3}(3)	0.5528	0.3250
{5}{3}(3)	-.1906	0.5103
{3}{3}(3)	-.2006	0.4107
{7}{5}(3)	0.0000	0.2249
{5}{5}(3)	0.0000	0.2249

2.20 $\{\rho\} = \{7\}, [\bar{f}] = [311], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]	(7)[322]
{6}2{5}(5)	0.9160	0.0000
{6}{5}(3)	0.0000	-.6222
{6}1{5}(5)	0.0000	-.3381
{6}{4}(4)	-.1138	-.5669
{6}{3}(3)	-.1713	0.0000
{4}{4}(4)	0.1533	-.4208
{4}{3}(3)	0.3083	0.0000

2.21 $\{\rho\} = \{7\}, [\bar{f}] = [221], [f_{12}] = [2]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[421]	(7)[322]
{7}{5}(5)	0.4629	-.1369
{5}{5}(5)	0.4629	0.1369
{7}{4}(4)	0.6547	0.4905
{5}{4}(4)	0.3780	-.8496

2.22 $\{\rho\} = \{7\}, [\bar{f}] = [221], [f_{12}] = [11]$

$(\upsilon)[f]$ / $\{\bar{\rho}\}\{\bar{\bar{\rho}}\}(\bar{\upsilon})$	(5)[331]	(7)[322]
{6}{4}(4)	-.5000	-.5078
{4}{4}(4)	-.8660	0.2932
{6}{5}(5)	0.0000	0.8101

Tables-3

Spin-Isospin FPC

3.1 $[f] = [43], \quad [\bar{\bar{f}}] = [41], \quad [f_{12}] = [2]$

$\bar{S}\bar{T}, \quad s\,t$ / $S\,T$	$\frac{1}{2}\frac{1}{2},10$	$\frac{1}{2}\frac{1}{2},01$
$\frac{1}{2}\,\frac{1}{2}$	-.3780	0.3780

3.2 $[f]=[43],\quad [\bar{\bar{f}}]=[32],\quad [f_{12}]=[2]$

$\bar{S}\bar{T},\ st$ / ST	$\frac{1}{2}\frac{1}{2},10$	$\frac{3}{2}\frac{1}{2},10$	$\frac{1}{2}\frac{1}{2},01$	$\frac{1}{2}\frac{3}{2},01$
$\frac{1}{2}\frac{1}{2}$	0.1890	0.3780	0.1890	0.3780

3.3 $[f]=[43],\quad [\bar{\bar{f}}]=[32],\quad [f_{12}]=[11]$

$\bar{S}\bar{T},\ st$ / ST	$\frac{1}{2}\frac{1}{2},00$	$\frac{1}{2}\frac{1}{2},11$	$\frac{3}{2}\frac{1}{2},11$	$\frac{1}{2}\frac{3}{2},11$
$\frac{1}{2}\frac{1}{2}$	0.1890	-.1890	0.3780	0.3780

3.4 $[f]=[421],\quad [\bar{\bar{f}}]=[41],\quad [f_{12}]=[2]$

$\bar{S}\bar{T},\ st$ / ST	$\frac{1}{2}\frac{1}{2},10$	$\frac{1}{2}\frac{1}{2},01$
$\frac{1}{2}\frac{1}{2}$	0.2390	0.2390
$\frac{3}{2}\frac{1}{2}$	-.3381	
$\frac{1}{2}\frac{3}{2}$		-.3381

3.5 $[f]=[421],[\bar{\bar{f}}]=[41],[f_{12}]=[11]$

$\bar{S}\bar{T},\ st$ / ST	$\frac{1}{2}\frac{1}{2},00$	$\frac{1}{2}\frac{1}{2},11$
$\frac{1}{2}\frac{1}{2}$	0.2390	-.2390
$\frac{3}{2}\frac{1}{2}$		-.3381
$\frac{1}{2}\frac{3}{2}$		-.3381

3.6 $[f]=[421],\quad [\bar{\bar{f}}]=[32],\quad [f_{12}]=[2]$

$\overline{S}\overline{T}$, st / ST	$\frac{1}{2}\frac{1}{2},10$	$\frac{3}{2}\frac{1}{2},10$	$\frac{1}{2}\frac{3}{2},10$	$\frac{1}{2}\frac{1}{2},01$	$\frac{3}{2}\frac{1}{2},01$	$\frac{1}{2}\frac{3}{2},01$
$\frac{1}{2}\ \frac{1}{2}$	-.1890	0.1890		0.1890		-.1890
$\frac{3}{2}\ \frac{1}{2}$	-.1336	0.2113			0.2835	
$\frac{1}{2}\ \frac{3}{2}$			-.2835	0.1336		-.2113

3.7 $[f]=[421],\quad [\bar{\bar{f}}]=[32],\quad [f_{12}]=[11]$

$\overline{S}\overline{T}$, st / ST	$\frac{3}{2}\frac{1}{2},00$	$\frac{1}{2}\frac{3}{2},00$	$\frac{1}{2}\frac{1}{2},11$	$\frac{3}{2}\frac{1}{2},11$	$\frac{1}{2}\frac{3}{2},11$
$\frac{1}{2}\ \frac{1}{2}$				0.2673	-.2673
$\frac{3}{2}\ \frac{1}{2}$	0.1336		-.1890	0.2988	
$\frac{1}{2}\ \frac{3}{2}$		-.1336	0.1890		-.2988

3.8 $[f]=[421],\quad [\bar{\bar{f}}]=[311],\quad [f_{12}]=[2]$

$\overline{S}\overline{T}$, st / ST	$\frac{1}{2}\frac{1}{2},10$	$\frac{3}{2}\frac{1}{2},10$	$\frac{1}{2}\frac{3}{2},10$	$\frac{3}{2}\frac{3}{2},10$	$\frac{1}{2}\frac{1}{2},01$	$\frac{3}{2}\frac{1}{2},01$	$\frac{1}{2}\frac{3}{2},01$	$\frac{3}{2}\frac{3}{2},01$
$\frac{1}{2}\ \frac{1}{2}$	-.1952	0.2182			-.1952		0.2182	
$\frac{3}{2}\ \frac{1}{2}$	-.0690	-.2440				0.1091		0.3086
$\frac{1}{2}\ \frac{3}{2}$			0.1091	0.3086	-.0690		-.2440	

3.9 $[f]=[421],\quad [\overline{\overline{f}}]=[311],\quad [f_{12}]=[11]$

$\overline{S}\rightleftarrows\overline{T}$, st / S T	$\frac{1}{2}\frac{1}{2}$,00	$\frac{3}{2}\frac{1}{2}$,00	$\frac{1}{2}\frac{3}{2}$,00	$\frac{1}{2}\frac{1}{2}$,11	$\frac{3}{2}\frac{1}{2}$,11	$\frac{1}{2}\frac{3}{2}$,11	$\frac{3}{2}\frac{3}{2}$,11
$\frac{1}{2}\ \frac{1}{2}$	-.1195			-.2789	-.0891	-.0891	-.2520
$\frac{3}{2}\ \frac{1}{2}$		-.1336		0.0282	0.0996	0.2520	0.2817
$\frac{1}{2}\ \frac{3}{2}$			-.1336	0.0282	0.2520	0.0996	0.2817

3.10 $[f]=[331],\quad [\overline{\overline{f}}]=[32],\quad [f_{12}]=[2]$

$\overline{S}\,\overline{T}$, st / S T	$\frac{1}{2}\frac{1}{2}$,10	$\frac{3}{2}\frac{1}{2}$,10	$\frac{1}{2}\frac{3}{2}$,10	$\frac{1}{2}\frac{1}{2}$,01	$\frac{3}{2}\frac{1}{2}$,01	$\frac{1}{2}\frac{3}{2}$,01
$\frac{1}{2}\ \frac{1}{2}$	-.3086	0.1543		-.3086		0.1543
$\frac{3}{2}\ \frac{1}{2}$	-.2440	-.3858			0.1725	
$\frac{1}{2}\ \frac{3}{2}$			0.1725	-.2440		-.3858
$\frac{3}{2}\ \frac{3}{2}$			-.3450		-.3450	

3.11 $[f]=[331],\quad [\overline{\overline{f}}]=[32],\quad [f_{12}]=[11]$

$\overline{S}\rightleftarrows\overline{T}$, st / S T	$\frac{1}{2}\frac{1}{2}$,00	$\frac{3}{2}\frac{1}{2}$,00	$\frac{1}{2}\frac{3}{2}$,00	$\frac{1}{2}\frac{1}{2}$,11	$\frac{3}{2}\frac{1}{2}$,11	$\frac{1}{2}\frac{3}{2}$,11
$\frac{1}{2}\ \frac{1}{2}$	-.1890			-.4410	-.0630	-.0630
$\frac{3}{2}\ \frac{1}{2}$		-.2113		0.0996	0.1575	0.3984
$\frac{1}{2}\ \frac{3}{2}$			-.2113	0.0996	0.3984	0.1575
$\frac{3}{2}\ \frac{3}{2}$				-.1992	-.3150	-.3150

3.12 $[f] = [322], \quad [\overline{f}] = [221], \quad [f_{12}] = [11]$

$\overline{S \rightleftarrows T}$, st / S T	$\frac{3}{2}\frac{1}{2}, 00$	$\frac{1}{2}\frac{3}{2}, 00$	$\frac{1}{2}\frac{1}{2}, 11$	$\frac{3}{2}\frac{1}{2}, 11$	$\frac{1}{2}\frac{3}{2}, 11$	$\frac{3}{2}\frac{3}{2}, 11$	$\frac{5}{2}\frac{1}{2}, 11$	$\frac{1}{2}\frac{5}{2}, 11$
$\frac{1}{2}\ \frac{1}{2}$				-.345	.345			
$\frac{3}{2}\ \frac{1}{2}$	.2315		.244	-.0345		-.3086	.169	
$\frac{1}{2}\ \frac{3}{2}$		-.2315	-.244		.0345	.3086		-.169
$\frac{3}{2}\ \frac{3}{2}$				.2182	-.2182		.2673	-.2673

3.13 $[f] = [322], \quad [\overline{f}] = [311], \quad [f_{12}] = [11]$

$\overline{S} \rightleftarrows \overline{T}$, st / S T	$\frac{1}{2}\frac{1}{2}, 00$	$\frac{3}{2}\frac{1}{2}, 00$	$\frac{1}{2}\frac{3}{2}, 00$	$\frac{3}{2}\frac{3}{2}, 00$	$\frac{1}{2}\frac{1}{2}, 11$	$\frac{3}{2}\frac{1}{2}, 11$	$\frac{1}{2}\frac{3}{2}, 11$	$\frac{3}{2}\frac{3}{2}, 11$
$\frac{1}{2}\ \frac{1}{2}$	.345				-.1150	-.2572	-.2572	.1455
$\frac{3}{2}\ \frac{1}{2}$		-.1725			.1818	.3343	-.3253	.0727
$\frac{1}{2}\ \frac{3}{2}$			-.1725		.1818	-.3253	.3343	.0727
$\frac{3}{2}\ \frac{3}{2}$				.2182	-.2300	.1626	.1626	-.3637

3.14 $[f] = [322], \quad [\overline{f}] = [32], \quad [f_{12}] = [2]$

$\overline{S} \rightleftarrows \overline{T}$, st / S T	$\frac{1}{2}\frac{1}{2}$, 10	$\frac{3}{2}\frac{1}{2}$, 10	$\frac{1}{2}\frac{3}{2}$, 10	$\frac{1}{2}\frac{1}{2}$, 01	$\frac{3}{2}\frac{1}{2}$, 01	$\frac{1}{2}\frac{3}{2}$, 01
$\frac{1}{2}\ \frac{1}{2}$	-.244	-.244		.244		.244
$\frac{3}{2}\ \frac{1}{2}$	.3858	-.122			.2728	
$\frac{1}{2}\ \frac{3}{2}$			-.2728	-.3858		.122
$\frac{3}{2}\ \frac{3}{2}$			.3450		-.3450	

3.15 $[f] = [331], \quad [\overline{\overline{f}}] = [311], \quad [f_{12}] = [2]$

$\overline{S}\,\overline{T}$, s t / S T	$\frac{1}{2}\frac{1}{2}$,10	$\frac{3}{2}\frac{1}{2}$,10	$\frac{1}{2}\frac{3}{2}$,10	$\frac{3}{2}\frac{3}{2}$,10	$\frac{1}{2}\frac{1}{2}$,01	$\frac{3}{2}\frac{1}{2}$,01	$\frac{1}{2}\frac{3}{2}$,01	$\frac{3}{2}\frac{3}{2}$,01
$\frac{1}{2}\ \frac{1}{2}$	-.1543	-.3450			.1543		.3450	
$\frac{3}{2}\ \frac{1}{2}$	.2440	-.1725				.3858		-.2182
$\frac{1}{2}\ \frac{3}{2}$			-.3858	.2182	-.2440		.1725	
$\frac{3}{2}\ \frac{3}{2}$			-.1543	.3450		.1543		-.3450

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